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Efficient methods in the search for periodic oscillations in dynamical systems☆

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ABSTRACT

Efficient methods in the search for the periodic oscillations of dynamical systems are described. Their application to the sixteenth Hilbert problem for quadratic systems and the Aizerman problem is considered. A synthesis of the method of harmonic linearization with the applied bifurcation theory and numerical methods for calculting periodic oscillations is described.

1. Introduction

In establishing and developing the theory of non-linear oscillations in the first half of the twentieth century,[1-4] most attention was given to analysing and synthesising oscillatory systems for which solving the problem of the existence of the oscillation modes did not present any great difficulties. The structure of many mechanical, electromechanical and electronic systems was such that there were oscillation modes in them, the existence of which was "almost obvious". The main attention of investigators was therefore concentrated on analysing of the form and properties of these ("almost" harmonic, relaxation, synchronous, circular, orbitally stable, etc.) oscillations.

In the 1950's, the attention of many scientists concentrated on two celebrated problems, the sixteenth Hilbert problem [5-8] and the Aizerman problem,[9–13] for which the proof of the existence of periodic solutions was not a trivial problem, and considerable progress was made in investigating these problems. It was found that they have a lot in common: while the problem of searching for periodic solutions in the case of two-dimensional periodic systems was formulated by Hilbert from the very beginning, it was revealed during investigations into the Aizerman problem that the differential equations of automatic control systems, satisfying the generalized Routh–Hurwitz conditions, can also have periodic solutions.[13] The problem of searching for the periodic solutions of such differential equations is also a current problem for subsequent investigations in this direction.

These two problems stimulated an enormous number of investigations in the second half of the twentieth century. Hilbert's sixteenth problem stimulated the development of bifurcation theory and the theory of normal modes, and the Aizerman problem stimulated theories of absolute stability. The most complete bibliography is available in Refs. 14–19, in which there are more than two thousand references. This review is dedicated to some efficient methods and techniques for searching for periodic solutions, which arose as the result of these investigations, and to both analytical and numerical methods. An attempt is made here to reflect the current trends in the synthesis and analytical and numerical methods, including powerful computer techniques for solving complex mathematical problems.

2. Cycles of two-dimensional quadratic systems

The Kolmogorov problem. Arnol'd writes:[20] "In order to estimate the number of limit cycles of quadratic vector fields in a plane, Kolmogorov distributed several hundreds of such fields (with randomly chosen coefficients of the second-degree polynomials) among several hundred students in the Mechanical-Mathematical Faculty of Moscow State University as a tutorial exercise. Each student had to find the number of limit cycles of his own field. The results of this experiment were completely unexpected: not a single limit cycle was found in any field!

A limit cycle is conserved when the field coefficients are slightly changed. Hence, systems with one, two, three (and even, as would become known later, four) limit cycles form open sets in the space of the coefficients such that the probabilities of entering into these sets in the case of a random choice of the polynomial coefficients are positive.

This fact that this did not happen suggests that the above-mentioned probabilities are obviousely small".

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The result of this experiment also demonstrates something else: the need to develop purposeful methods for searching for periodic oscillations, that is, both analytical and numerical methods, which make use of the full power of current computational techniques. This review is concerned with describing some of these methods.

Here, we are concerned with the Kolmogorov problem and we will elucidate whether two-dimensional quadratic dynamical systems exist for which the students might have revealed limit cycles in their tutorial exercise described above. To do this, we reduce an arbitrary quadratic system to the special Liénard equation. For quadratic systems, transformations to the Liénard equation have been used in Refs. 21-28. Here, we follow Refs. 26-28.

We consider the quadratic system

$$\dot{x} = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y$$
$$\dot{y} = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y \tag{2.1}$$

where $a_i, b_i, c_i, \alpha_i, \beta_i$ are real numbers.

Assumption 1. Without loss of generality, it can be assumed that $c_1 = 0$.

Proof. To be specific, we will assume that $a_2 \neq 0$ (otherwise, on changing the notation $x \to y, y \to x$, we immediately obtain $c_1 = 0$). We next introduce the linear transform $x_1 = x + \nu y, y_1 = y$. To prove Assumption 1, it is sufficient to show that the identity $(x + \nu y) = (a_1 + \nu a_2)x^2 + (b_1 + \nu b_2)xy + (c_1 + \nu c_2)y^2 + (\alpha_1 + \nu\alpha_2)x + (\beta_1 + \nu\beta_2)y = \rho(x + \nu y)y + k(x + \nu y)^2 + (\alpha_1 + \nu\alpha_2)x + (\beta_1 + \nu\beta_2)y$, which is equivalent to the following system of equations

$$\kappa = a_1 + \nu a_2, \quad \kappa\nu^2 + \rho\nu = c_1 + \nu c_2, \quad \rho + 2\kappa\nu = b_1 + \nu b_2 \tag{2.2}$$

holds for certain numbers ρ, κ and ν.

Equalities (2.2) are satisfied if

$$(a_1 + \nu a_2)\nu^2 - \nu(b_1 + \nu b_2) + (c_1 + \nu c_2) = 0$$

Since $a_2 \neq 0$, this equation of the third degree in ν always has a real root. Hence, the system of equations (2.2) always has a real solution. We will henceforth assume that $c_1 = 0$.

Assumption 2. Suppose $b_1 \neq 0$. The straight line $\beta_1 + b_1x = 0$ in the plane $\{x, y\}$ is either invariant or transversal for trajectory system (2.1).

Proof. This assertion follows from the equality

$$(\beta_1 + b_1x)^\bullet = b_1[(\beta_1 + b_1x)y + a_1x^2 + \alpha_1x] = [a_1(\beta_1/b_1)^2 - \alpha_1(\beta_1/b_1)]b_1$$

when $x = -\beta_1/b_1$: if the expression in the square brackets is equal to zero, the straight line $\beta_1 + b_1x = 0$ is invariant, and if it is not equal to zero, this straight line is transversal.

Next, excluding the trivial case, when the right-hand side of the first equation of (2.1) is independent of y, from the treatment, we shall assume that

$$|b_1| + |\beta_1| \neq 0 \tag{2.3}$$

It follows from this and from Assumption 2 that the limiting cycles of system (2.1) are also trajectories of the system

$$\dot{x} = y + \frac{a_1x^2 + \alpha_1x}{\beta_1 + b_1x}, \quad \dot{y} = \frac{a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y}{\beta_1 + b_1x} \tag{2.4}$$

We now introduce the transformation

$$\bar{y} = y + \frac{a_1x^2 + \alpha_1x}{\beta_1 + b_1x}, \quad \bar{x} = x$$

into the treatment.

In these new phase variables (we will subsequently omit the dashes over the variables: $\bar{x} \to x, \bar{y} \to y$) system (2.4) is written in the form

$$\dot{x} = y, \quad \dot{y} = -Q(x)y^2 - R(x)y - P(x) \tag{2.5}$$

where

$$Q(x) = \frac{-c_2}{\beta_1 + b_1 x}$$

$$R(x) = -\frac{(b_1 b_2 - 2a_1 c_2 + a_1 b_1)x^2 + (b_2\beta_1 + b_1\beta_2 - 2\alpha_1 c_2 + 2a_1\beta_1)x + \alpha_1\beta_1 + \beta_1\beta_2}{(\beta_1 + b_1 x)^2}$$

$$P(x) = -\left(\frac{a_2x^2 + \alpha_2 x}{\beta_1 + b_1 x} - \frac{(b_2 x + \beta_2)(a_1 x^2 + \alpha_1 x)}{(\beta_1 + b_1 x)^2} + \frac{c_2(a_1 x^2 + \alpha_1 x)^2}{(\beta_1 + b_1 x)^3}\right)$$

It follows from Assumption 2 and condition (2.3) that the trajectories of system (2.5) are also trajectories of the system

$$\dot{x} = ye^{p(x)}, \quad \dot{y} = [-Q(x)y^2 - R(x)y - P(x)]e^{p(x)}$$

where $p(x)$ is a certain integral of the function $Q(x)$.

Using the substitution $\bar{x} = x$, $\bar{y} = ye^{p(x)}$, from this system we obtain the system

$$\dot{x} = y, \quad \dot{y} = -f(x)y - g(x) \tag{2.6}$$

Here, after the above-mentioned transformation, the dashes over the variables x and y are again omitted.

Hence, when investigating the limit cycles, system (2.1) when $b_1 \neq 0$ can be reduced, using the above-mentioned substitutions, to the Liénard equations (2.6) with the functions

$$f(x) = R(x)e^{p(x)} = R(x)|\beta_1 + b_1 x|^q,$$

$$g(x) = P(x)e^{2p(x)} = P(x)|\beta_1 + b_1 x|^{2q}; \quad q = -c_2/b_1$$

and, when $b_1 = 0$, to Eq. (2.6) with the functions

$$f(x) = R(x)e^{qx}, \quad g(x) = P(x)e^{2qx}; \quad q = -c_2/\beta_1$$

When $b_1 \neq 0$, the limit cycle of system (2.1) passes into the limit cycle of system (2.6), which is located to the right or to the left of the line of discontinuity $\beta_1 + b_1 x = 0$, $y \in R^1$. This follows from Assumption 2.

Liénard's theorem on the existence of a limit cycle is well known in the case of Eq. (2.6) with smooth functions f and g.[29,30] Here, we will present an extension of this theorem to the case of discontinuous functions f and g and apply it to Kolmogorov's problem. To do this, we shall assume that the functions $f(x)$ and $g(x)$ are differentiable in the interval $(a, +\infty)$ and that the following conditions are satisfied for certain numbers $a < \nu_1 \le x_0 \le \nu_2$

1) $g(x) < 0, \quad \forall x \in (a, x_0); \quad g(x) > 0, \quad \forall x \in (x_0, +\infty)$

$$\lim_{x\to a} G(x) = \lim_{x\to+\infty} G(x) = +\infty; \quad G(x) = \int_{x_0}^{x} g(z)dz$$

2) $f(x) > 0, \quad \forall x \in (a, \nu_1) \cup (\nu_2, +\infty); \quad F_1(\nu_2) \ge 0; \quad F_k(x) = \int_{\nu_k}^{x} f(z)dz,$

$k = 1, 2$

Theorem 1. *Suppose Conditions 1 and 2 are satisfied and the point $x = x_0$, $y = 0$ is an unstable focus equilibrium state in the Lyapunov sense. System (2.6) then has a limit cycle.*

Proof. Consider a pair of numbers $\mu_1 \in (a, \nu_1)$ and $\mu_2 \in (\nu_2, +\infty)$ such that the number μ_1 is sufficiently close to a and the number μ_2 is sufficiently large, and

$$\int_{\mu_1}^{\mu_2} g(z)dz = 0 \tag{2.7}$$

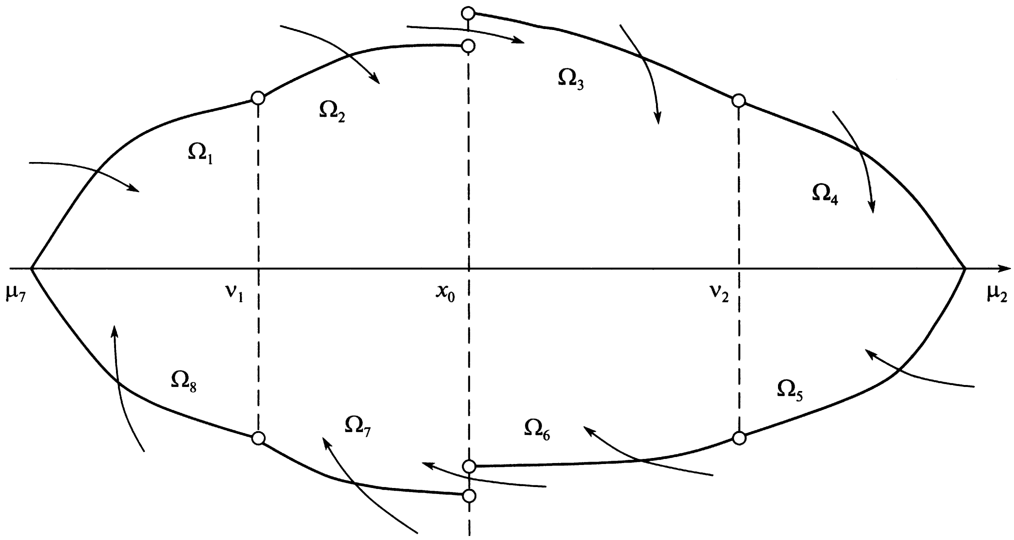

Fig. 1.

We now introduce the functions

$$V_1(x, y) = y^2 + 2G(x)$$
$$V_{k+1}(x, y) = (y + F_k(x))^2 + 2G(x)$$
$$V_{k+3}(x, y) = V_{k+1}(x, y) - \varepsilon(x - \nu_k)$$
$$V_{k+5}(x, y) = V_{4-k}(x, y) + \varepsilon(x - \nu_{3-k}); \quad k = 1, 2$$

into the treatment. Here, ε is a certain sufficiently small number.

We now define the sets Ω_j in the following manner (Fig. 1)

$$\Omega_1 = \{x \in [\mu_1, \nu_1],\ y > 0,\ V_1(x, y) = V_1(\mu_1, 0)\}$$
$$\Omega_2 = \{x \in [\nu_1, x_0],\ y > 0,\ V_4(x, y) = V_2(\nu_1, y_1)\}$$
$$\Omega_3 = \{x \in [x_0, \nu_2],\ y > 0,\ V_5(x, y) = V_3(\nu_2, y_2)\}$$
$$\Omega_4 = \{x \in [\nu_2, \mu_2],\ y > 0,\ V_3(x, y) = V_3(\mu_2, 0)\}$$
$$\Omega_5 = \{x \in [\nu_2, \mu_2],\ y < 0,\ V_1(x, y) = V_1(\mu_2, 0)\}$$
$$\Omega_6 = \{x \in [x_0, \nu_2],\ y < 0,\ V_6(x, y) = V_3(\nu_2, y_3)\}$$
$$\Omega_7 = \{x \in [\nu_1, x_0],\ y < 0,\ V_7(x, y) = V_2(\nu_1, y_4)\}$$
$$\Omega_8 = \{x \in [\mu_1, \nu_1],\ y < 0,\ V_2(x, y) = V_2(\mu_1, 0)\}$$

Here, $y_0 > 0, y_2 > 0, y_3 < 0, y_4 < 0$ are the solutions of the quadratic equations

$$V_1(\nu_1, y_1) = V_1(\mu_1, 0), \quad V_3(\nu_2, y_2) = V_3(\mu_2, 0),$$
$$V_1(\nu_2, y_3) = V_1(\mu_2, 0), \quad V_2(\nu_1, y_4) = V_2(\mu_1, 0)$$

It is clear that, in the case of arbitrary functions $V_j(x, y)$, the relations

$$\dot{V}_1 = -f(x)y^2$$
$$\dot{V}_{k+1} = -2g(x)F_k(x), \quad \dot{V}_{k+3} = -2g(x)F_k(x) - \varepsilon y,$$
$$\dot{V}_{k+5} = -2g(x)F_{3-k}(x) + \varepsilon y; \quad k = 1, 2 \tag{2.8}$$

hold along the trajectories of system (2.6).

From these relations, it immediately follows that $\dot{V}_1 < 0$ in $\Omega_1 \cup \Omega_5$, $\dot{V}_2 < 0$ in Ω_8 and $\dot{V}_3 < 0$ in Ω_4. The trajectories of system (2.6) therefore intersect these sets as shown in Fig. 1.

We will now show that the relations

$$\dot{V}_4 < 0, \quad \dot{V}_5 < 0, \quad \dot{V}_6 < 0, \quad \dot{V}_7 < 0 \tag{2.9}$$

are also satisfied in the sets Ω_2, Ω_3, Ω_6, Ω_7 respectively for a specific choice of the parameters ε, μ_1 and μ_2. In fact, having fixed $\varepsilon > 0$, we choose μ_1 and μ_2 to be so close to a and $+\infty$ that the minimum value of y in the sets Ω_2, Ω_3, Ω_6, Ω_7 will be greater than

$$\frac{1}{\varepsilon} \max_{x \in [\nu_1, \nu_2]} 2|g(x)F_k(x)|$$

Such a choice of μ_1 and μ_2 is possible by virtue of Conditions 1. Inequalities (2.9) follow from this and relations (2.8).

Note (see Fig. 1) that, from conditions 2 and (2.7) for sufficiently small ε, we have the inequalities $y_5 < y_6$ and $y_7 > y_8$, where $y_{k+4} (k = 1, 2)$ is a positive solution of the equation

$$V_{k+3}(x_0, y_{k+4}) = V_{k+1}(\nu_k, y_k), \quad k = 1, 2 \tag{2.10}$$

and $y_{k+6} (k = 1, 2)$ is a negative solution of the equation

$$V_{k+5}(x_0, y_{k+6}) = V_{4-k}(\nu_{3-k}, y_{k+2}), \quad k = 1, 2 \tag{2.11}$$

In fact, we initially put $\varepsilon = 0$, and the equalities (2.10) and (2.11) then take the form

$$\begin{aligned} &(y_{k+4} + F_k(x_0))^2 = y_k^2 + 2G(\nu_1), \\ &(y_{k+6} + F_{3-k}(x_0))^2 = y_{k+2}^2 + 2G(\nu_{3-k}); \quad k = 1, 2 \end{aligned} \tag{2.12}$$

Since

$$y_1^2 = 2G_1, \quad y_2^2 = 2G_2 + F_2^2(\mu_2); \quad y_3^2 = 2G_2, \quad y_4^2 = 2G_1 + F_1^2(\mu_1);$$

$$G_k = \int_{\nu_k}^{\mu_k} g(z)dz, \quad k = 1, 2$$

the equalities

$$(y_5 + F_1(x_0))^2 = (y_6 + F_2(x_0))^2 - F_2^2(\mu_2), \quad (y_7 + F_2(x_0))^2 = (y_8 + F_1(x_0))^2 - F_1^2(\mu_1)$$

follow from relations (2.12) and condition (2.7), and the estimates

$$y_5 < y_6 + F_2(\nu_1), \quad y_7 > y_8 + F_1(\nu_2)$$

follow from these.

When $\varepsilon = 0$, the inequalities $y_5 < y_6$ and $y_7 > y_8$ follow from this and from Condition 2. It is clear that these inequalities remain true in the case of small ε.

Hence, a closed transversal curve has been constructed here, which is shown in Fig. 1. Since the unique equilibrium state in the $\{x > a\}$ half plane, $x = x_0$, $y = 0$, is an unstable Lyapunov focus, we obtain from this, using the well known ring principle (Fig. 2), that system (2.6) has a cycle.

The following simple assertions will be useful later.

Assumption 3. Suppose $c_1 = 0$, $\beta_1 \neq 0$. Then, without loss of generality, it can be assumed that $\alpha_1 = 0$.

This assertion is proved using the linear substitution $\mu\beta * q \leq \alpha\omega_0^2$.

Assumption 4. Suppose $c_1 = 0$, $\alpha_1 = 0$, $a_1 \neq 0$, $b_1 \neq 0$, $\beta_1 \neq 0$. Then, without loss of generality, it can be assumed that

$$c_1 = 0, \quad \alpha_1 = 0, \quad a_1 = b_1 = \beta_1 = 1 \tag{2.13}$$

This assertion is proved using the linear substitution

$$x = \frac{\beta_1}{b_1} x_1, \quad y = \frac{a_1\beta_1}{b_1^2} y_1, \quad t = \frac{b_1}{a_1\beta_1} t_1$$

It follows from Assumptions 1–4 that almost any system (2.1) can be reduced by means of a linear transformation to a form such that relations (2.13) are satisfied.

If they are satisfied, Conditions 1 and 2 of Theorem 1 can be written in the form

$$c_2 \in (0, 1), \quad b_2 - c_2 > a_2, \quad 2c_2 > b_2 + 1 \tag{2.14}$$

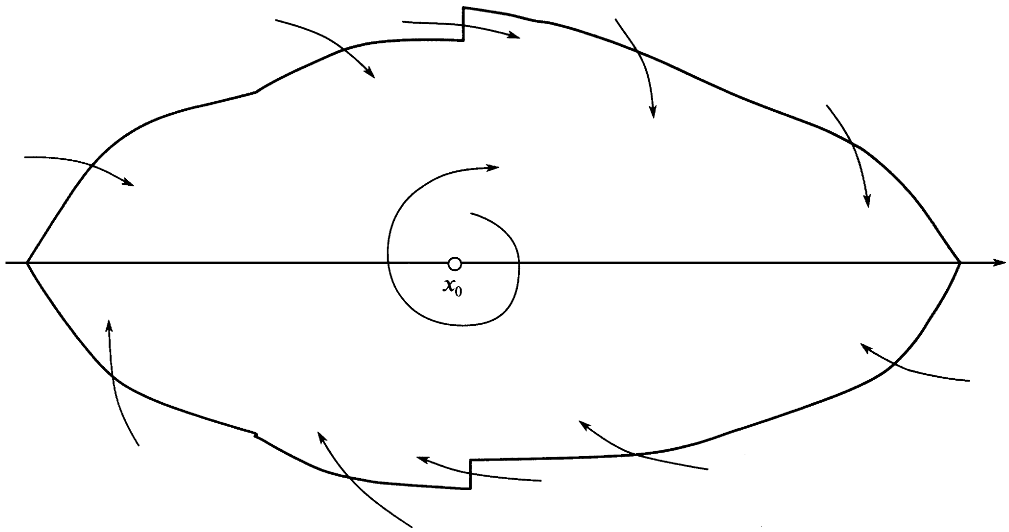

Fig. 2.

Moreover, it is necessary that the polynomial

$$(a_2x+\alpha_2)(x+1)^2-(b_2x+\beta_2)x(x+1)+c_2x^3 \tag{2.15}$$

should not have real roots in the interval $(-1, +\infty)$.

With the condition that $x=y=0$ is an unstable focus, the inequalities here will be

$$\alpha_2<-\beta_2^2/4, \quad \beta_2>0 \tag{2.16}$$

It follows from conditions (2.14) that polynomial (2.15) necessarily has a real root in the interval (-4, -1) Hence, in order that this polynomial does not have a real root in the interval $(-1, +\infty)$, it is sufficient that this real root is unique.

We rewrite the polynomial (2.15) in the form

$$ax^3+bx^2+cx+d$$

$$a = a_2-b_2+c_2, \quad b = \alpha_2+2a_2-b_2-\beta_2, \quad c = 2\alpha_2+a_2-\beta_2, \quad d = \alpha_2$$

It is well known [31] that the inequality

$$\Delta = 4c^3a-c^2b^2-18abcd+27a^2d^2+4b^3d>0$$

is the condition for the real root of this polynomial to be unique

Its left-hand side can be written in the form of a third degree polynomial in α_2:

$$\begin{aligned}\Delta(\alpha_2) &= -4c_2\alpha_2^3+(-\beta_2^2+(2b_2+6c_2)\beta_2+27c_2^2+(12a_2-18b_2)c_2-b_2^2)\alpha_2^2\\ &-2(-\beta_2^3+(-3c_2-\alpha_2+4b_2)\beta_2^2+(-5b_2^2+9b_2c_2+2b_2a_2-3a_2c_2)\beta_2\\ &-a_2b_2+2b_2^3-9a_2b_2c_2+6a_2^2c_2)\alpha_2-\beta_2^4+(2b_2-4c_2+2a_2)\beta_2^3\\ &+(12a_2c_2-b_2^2-a_2^2-4a_2b_2)\beta_2^2+(2a_2^2b_2-12a_2^2c_2+2a_2b_2^2)\beta_2-a_2b_2^2+4a_2^3c_2\end{aligned}$$

It is clear that the inequality $\Delta(\alpha_2)>0$ is satisfied in the case of

$$\alpha_2<\lambda(a_2, b_2, c_2, \beta_2) \tag{2.17}$$

where λ is the minimum root of the equation $\Delta(\lambda)=0$.

Hence, if relations (2.13), (2.14), (2.16) and (2.17) are satisfied, then all the conditions of Theorem 1 are satisfied and, consequently, system (2.1) has a limit cycle.

We now present some numerical examples.

Suppose inequalities (2.13) are satisfied and $a_2=-1$, $b_2=0$, $c_2=3/4$, $\beta_2=1$. Then,

$$\Delta = -3\alpha_2^3+\frac{155}{16}\alpha_2^2-9\alpha_2-28, \quad \lambda\approx-1.156$$

In this case, conditions (2.14), (2.16) and (2.17) are satisfied when $a_2<-1.2$. The limit cycles for $\alpha_2=-2-10, -100, -1000$ are shown in Fig. 3.

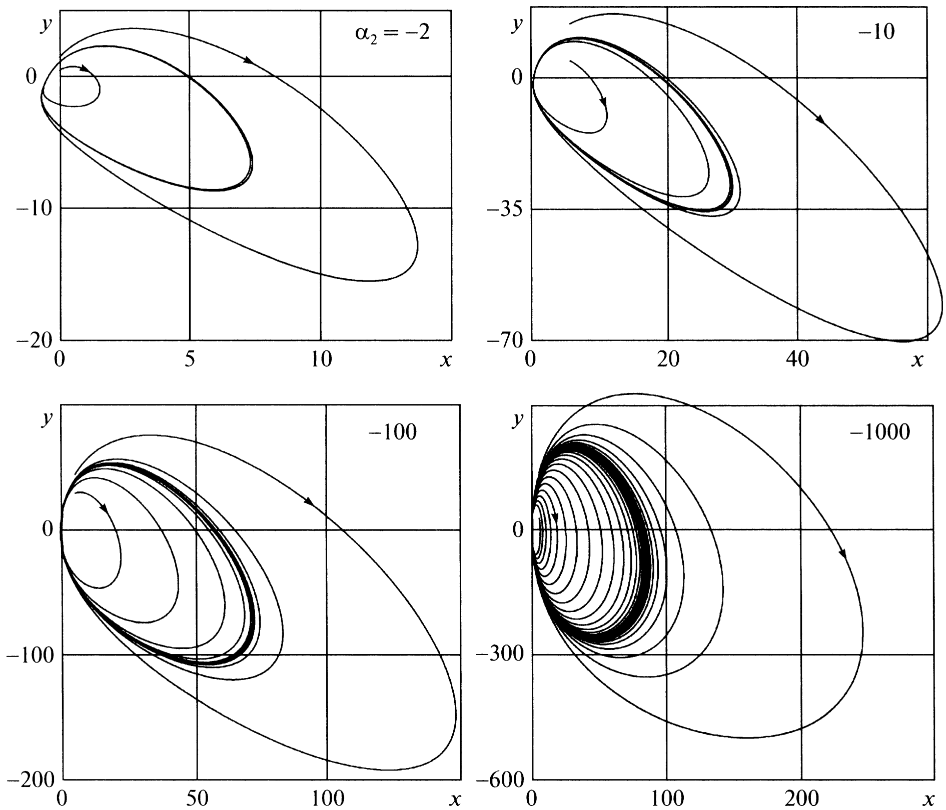

Fig. 3.

Suppose inequalities (2.13) are satisfied and $a_2 = -2, b_2 = -1/2, c_2 = 1/2, \beta_2 = 2$. Then,

$$\Delta = -2\alpha_2^3 - \alpha_2^2 + \frac{13}{2}\alpha_2 - 228, \quad \lambda \approx -5.2519$$

In this case, conditions (2.14), (2.16) and (2.17) are satisfied when $\alpha_2 < -6$. The limit cycles are shown in Fig. 4 for $\alpha_2 = -10, -100, -1000, -1500$.

Suppose inequalities (2.13) are satisfied and $a_2 = -5; b_2 = -1; c_2 = 1/4; \beta_2 = 3$. Then,

$$\Delta = -\alpha_2^3 - \frac{325}{16}\alpha_2^2 - 48\alpha_2 - 1536, \quad \lambda \approx -21.420$$

In this case, conditions (2.14), (2.16) and (2.17) are satisfied when $\beta_2 < -2$. The limit cycles are shown in Fig. 5 for $\alpha_2 = -25, -100, -1000, -1500$.

Finally, a student would have obtained these results if Kolmogorov had issued the problem to him with these parameters. Here, the limit cycles are "highly visible". They are found by virtue of the following purposeful actions. The conditions for the existence of globally stable limit cycles are well known for different type of Liénard equations describing the dynamics of mechanical, electromechanical and electronic systems.[29,30,33] Only when it became clear that an arbitrary quadratic system reduces to a special Liénard equation did the attempt to generalize[28,34] the previous classical investigations[29,30,32,33] to such a Liénard equation become the next natural step. This generalization also enabled one to obtain the conditions for the existence of limit cycles which separate out the set of infinite Lebesque measure in the parameter space of system (2.1). This set is not "small".

3. Domain in the parameter space of quadratic systems where four cycles exist

Kolmogorov certainly knew about Bautin's investigations.[5,35] He showed that up to three small cycles can exist in the neighbourhood of the zero equilibrium state of system (2.1). To do this, Bautin considered a so-called weak focus where $\alpha_1 = \beta_2 = 0$, $\beta_1 = 1$, $\alpha_2 = -1$ and the Lyapunov quantities [36] $L_1 = L_2 = 0$, $L_3 > 0$. Then, by slightly perturbing the coefficients, it is possible to achieve that the inequalities

$$L_2 < 0, \quad L_1 > 0, \quad \alpha_1 + \beta_2 < 0$$
$$L_3 \gg |L_2| \gg |L_1| \gg |\alpha_1 + \beta_2| \tag{3.1}$$

are satisfied.

Hence, the qualitative pattern of the behaviour of the trajectories, shown in Fig. 6, arises in the neighbourhood of the zero equilibrium.

It follows from this that, in this case, three limit cycles exist in the neighbourhood of the point $x = y = 0$. However, they are small and "almost invisible" with numerical integration of the trajectories even using current computational techniques.

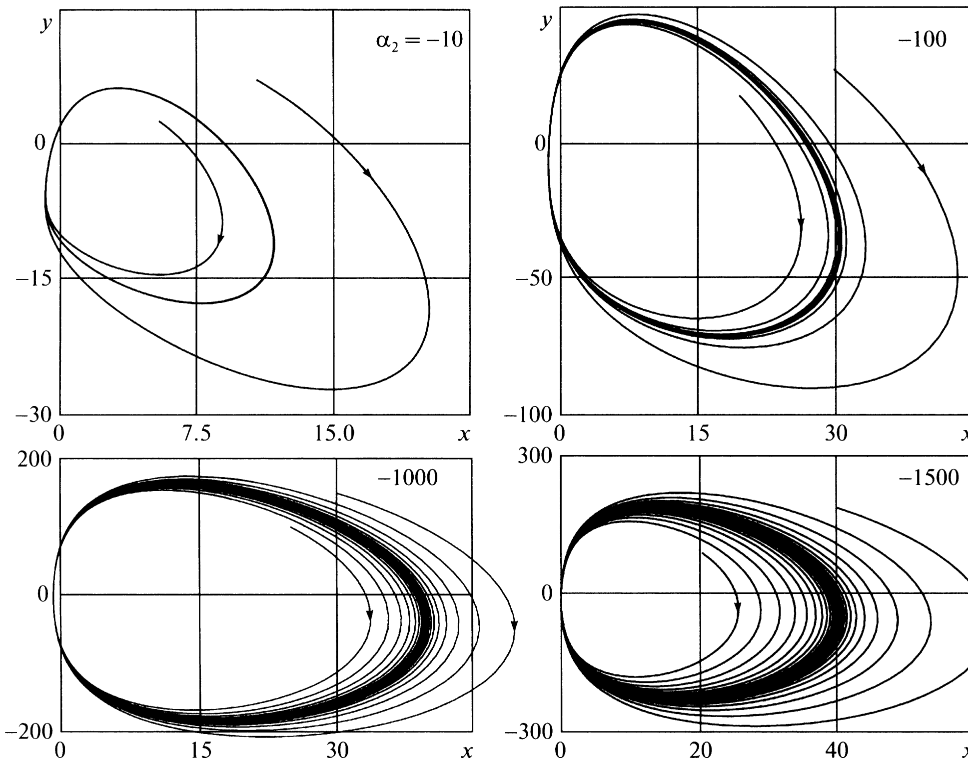

Fig. 4.

The first [37], second [38] and third [39] Lyapunov quantities have been calculated in the general case in the neighbourhood of the equilibrium state for two-dimensional systems with analytic right-hand sides. The corresponding formulae are considerably simplified in the case of the Liénard equation. We will present them, following the results obtained earlier [39]

Suppose, for system (2.6),

$$f(x) = f_0 + f_1 x + \ldots + f_6 x^6 + O(x^7)$$
$$g(x) = x + g_2 x^2 + \ldots + g_6 x^6 + O(x^7)$$

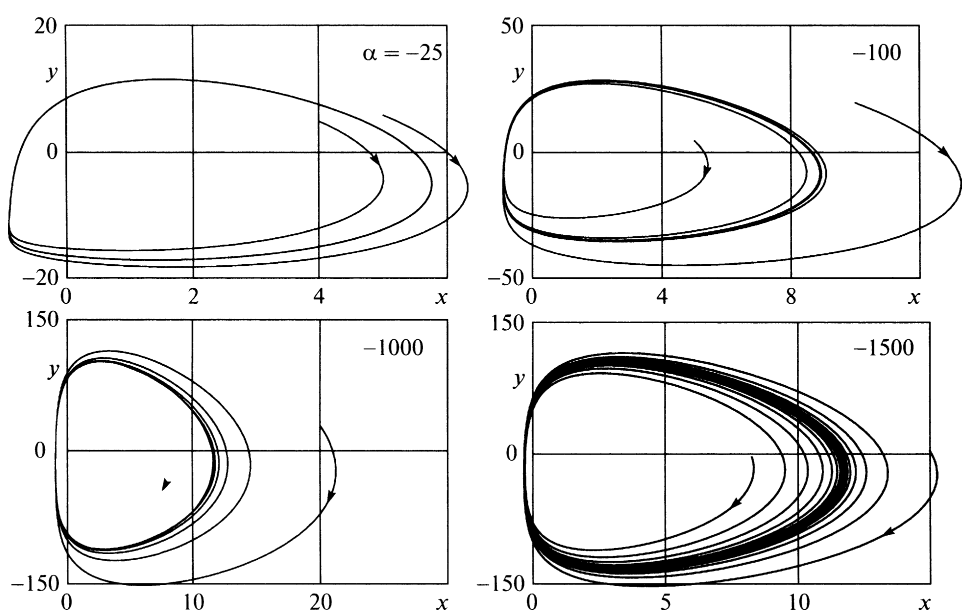

Fig. 5.

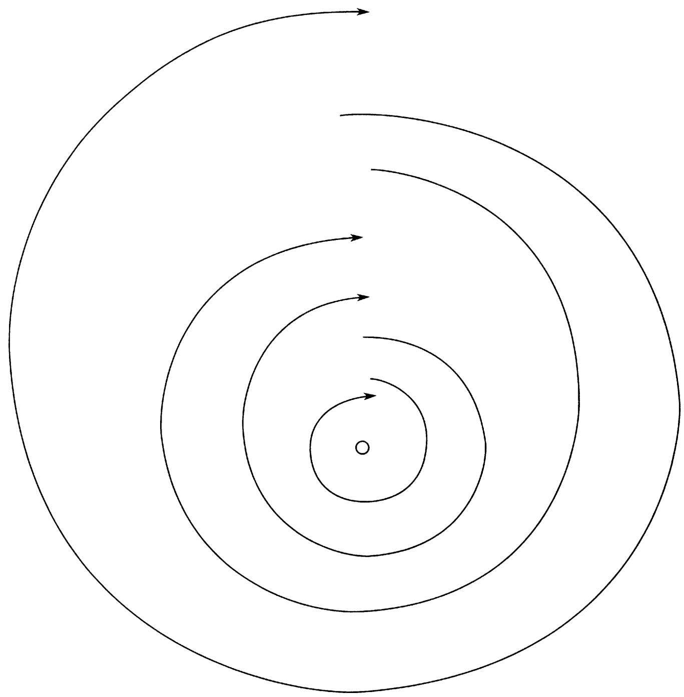

Fig. 6.

Then, if $f_0 = 0$,

$$L_1 = -\frac{\pi}{4}(g_2 f_1 - f_2)$$

If $f_2 = g_2 f_1$, then $L_1 = 0$ and

$$L_2 = \frac{\pi}{24}(3f_4 - 5g_2(f_3 - g_3 f_1) - 3g_4 f_1)$$

If $3f_4 = 5g_2(f_3 - g_3 f_1) + 3g_4 f_1$, then $L_1 = 0$ and

$$L_3 = -\frac{\pi}{576}(-45f_6 + 105g_2(f_5 - g_3 f_3 + g_3^2 f_1 - g_5 f_1) - 70g_2^3(f_3 - g_3 f_1)$$
$$- 63g_3 g_4 f_1 + 63g_4 f_3 + 45g_6 f_1)$$

Calculations of the Lyapunov quantities for various special forms of the Liénard system are also available in Ref. 17.

Since any quadratic system can be reduced to Liénard's equation, for the proof of the existence of the three small limit cycles in the neighbourhood of the point $x = y = 0$ it is sufficient to make use of the formulae for L_1, L_2 and L_3 presented above.

In order that a certain quadratic system (2.1) should correspond to an arbitrary Liénard system with the functions

$$f(x) = (Ax + B)x|x + 1|^{q-2}$$
$$g(x) = (C_1 x^3 + C_2 x^2 + C_3 x + 1)x\frac{|x + 1|^{2q}}{(x + 1)^3} \tag{3.2}$$

where A, B, C_j and q are certain numbers and $q \neq 1/2$, it is necessary and sufficient that the relations [34,39,40]

$$\frac{B - A}{(2q - 1)^2}((1 - q)B + (3q - 2)A) = 2C_2 - 3C_1 - C_3$$
$$\frac{B - A}{(2q - 1)^2}(B + 2(q - 1)A) = C_2 - 2C_1 - 1 \tag{3.3}$$

are satisfied.

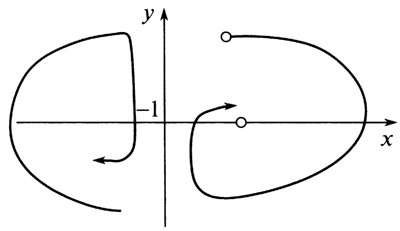

Fig. 7.

In this case, the parameters of system (2.1) are calculated in the following manner

$$b_1 = \beta_1 = \alpha_1 = 1, \quad \beta_2 = -1, \quad c_1 = 0, \quad c_2 = -q$$

$$a_1 = 1 + \frac{B-A}{2q-1}, \quad a_2 = -(q+1)a_1^2 - Aa_1 - C_1$$

$$b_2 = -A - a_1(2q+1), \quad \alpha_2 = -2$$

Hence, in the case when $x = y = 0$ is a weak focus, $c_1 = 0$, $b_1 \neq 0$, $2c_2 + b_1 \neq 0$ and, in searching for the limit cycles, it is possible to pass from system (2.1) to system (2.6) with functions of the form of (3.2) when relations (3.3) are satisfied.

Following the procedure described earlier [34,39], we obtain that relations (3.3) and $L_1 = L_2 = 0$ are satisfied if

$$A = \frac{1}{5}(2Bq + 4B)$$

$$C_1 = (q+3)\frac{B^2}{25} - \frac{(1+3q)}{5}, \quad C_2 = \frac{1}{25}(15(1-2q) + 3B^2), \quad C_3 = \frac{3(3-q)}{5} \tag{3.4}$$

In this case,

$$L_3 = L_3(B, q) = -\frac{\pi B(q+2)(3q+1)}{20000}(5(q+1)(2q-1)^2 + B^2(q-3)) \tag{3.5}$$

Hence, for arbitrary B and q which are such that $L_3(B, q) \neq 0$, it is possible to choose A, C_1C_2, C_3 such that inequalities (3.4) are satisfied. Then, by slightly perturbing the system parameters, we succeed, when $L_3(B, q) > 0$, in satisfying relations (3.1) and (3.2) and, when $L_3(B, q) < 0$, relations (3.2) and

$$L_2 > 0, \quad L_1 < 0, \quad \alpha_1 + \beta_2 > 0 \tag{3.6}$$

It has already been mentioned that, in this case, three cycles are generated in the neighbourhood of the point $x = y = 0$.

Remark. In passing from system (2.1) to system (2.6) and back again at the point $x = y = 0$, the signs and orders of smallness of the Lyapunov quantities are preserved. Hence, considering system (2.6) with $f(x)$ and $g(x)$ of the form of (3.2), for which relations (3.3) and (3.4) are satisfied, we obtain the existence of a certain system (2.1) for which

$$\alpha_1 + \beta_2 = 0, \quad L_1 = L_2 = 0$$

and, if $L_3(B, q) > 0$, then $L_3 > 0$ for system (2.1) also. Further, by choosing the small disturbances of system (2.1) in a special manner,[39] we obtain inequalities (3.1) (and, if $L_3(B, q) < 0$, inequalities (3.6)).

One of the first papers, where four limit cycles were revealed, was the paper by Shi.[41] A method for the asymptotic integration of the trajectories of Liénard's equation was developed [42] and, using this, Shi's results were generalized.

Theorem 2 ([42]). *Suppose relations (3.4), $B < 0$, $q \in (-1, -1/3)$ and*

$$B^2 < -5(q+1)(3q+1) \tag{3.7}$$

are satisfied in the case of system (2.6) with functions f and g of the form (3.2).

The behaviour of the trajectories of system (2.6) with sufficiently large initial data:

$$|x(0)| + |y(0)| \gg 1, \quad x(0) \neq -1$$

will then be as shown in Fig. 7. In this case, $L_3 < 0$, and the unique equilibrium state, located to the left of the line of discontinuity $\{x = -1, y \in R^1\}$, is unstable.

The following result follows directly from this.

Theorem 3 ([42]). *If relations (3.4), $B \neq 0$, $q \in (-1, -1/3)$ and (3.7) are satisfied, system (2.6), (3.2) has a limit cycle located to the left of the line of discontinuity $\{x = -1, y \in R^1\}$.*

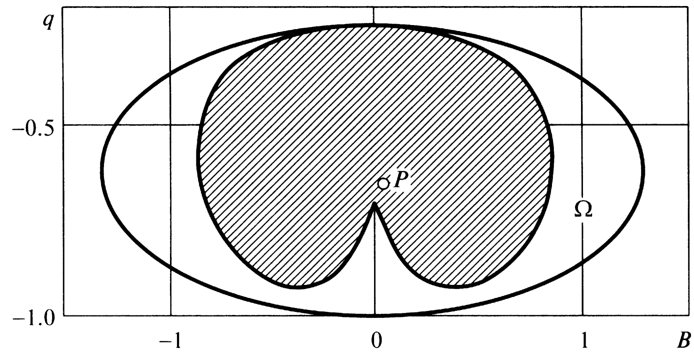

Fig. 8.

Theorem 3 and the above-mentioned weak perturbations of the parameters therefore enable one to separate out the classes of quadratic systems with four limit cycles: with one "large" cycle, for which $x(t) < -1 \forall t \in R^1$, and three small limit cycles located in the neighbourhood of the zero equilibrium.

The domain Ω, for which the conditions of Theorem 3 are satisfied, is shown in Fig. 8. The hatched domain is the set of the parameters B and q where the existence of four cycles (after small perturbations of the parameters) was obtained by Shi.[41] Note that this domain is contained as a whole in Ω.

The "large" stable cycles in the domain Ω (when $B < 0$) were calculated [39] by numerical integration of the trajectories.

As an example, we will consider the point P in Fig. 8 and transform the parameters B and q corresponding to it and the parameters from relations (3.4) into the parameters of system (2.1) As a result, we obtain the system

$$\dot{x} = 0.99x^2 + xy + y$$

$$\dot{y} = -0.58x^2 + 0.17xy + 0.6y^2 - 2x - y$$

The location of the trajectories tending to the "large" cycle when $t \to -\infty$ are shown for it in Fig. 9.

In computer calculations of the trajectories of systems (2.1) and (2.6) with a degenerate focus, we encounter with the effect of a flattening of the trajectories, which is clearly observed in Fig. 9 when the trajectories pass into a certain neighbourhood of the null point. Here, the trajectories "almost adhere". It can be said that here, unlike the widely known parametric stiffness of the system, trajectory stiffness is observed when the distance between the trajectories is an oscillating function of time and the values of this function at various points of the period of the oscillation differ by several orders of magnitude.

We will now present a further analogous example [39] of such trajectory stiffness for system (2.6) with

$$f(x) = \frac{(72x + 60)x}{x + 1}, \quad g(x) = \left(\frac{2876}{5}x^3 + \frac{2175}{5}x^3 + \frac{6}{5}x + 1\right)\frac{x}{x + 1}$$

Here also, $L_1 = L_2 = 0$, $L_3 \neq 0$. The trajectories shown in Fig. 10 are not periodic and approach very close to the zero stationary point in the region of the flattening. The distance between these the flattening region and the zero point reaches a value of 0.026.

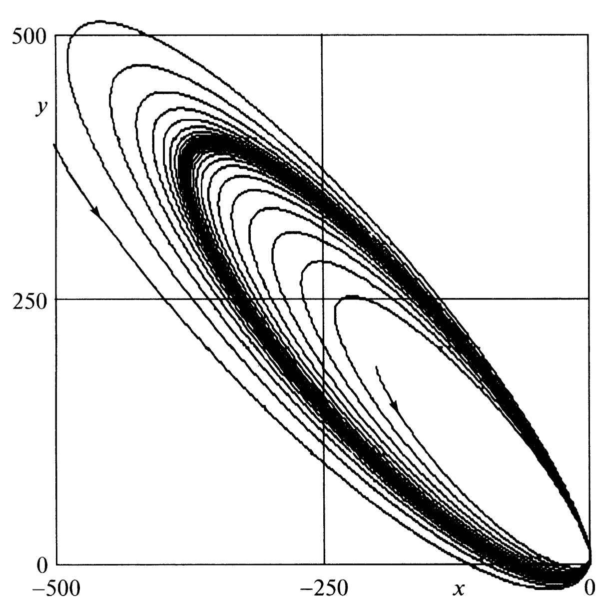

Fig. 9.

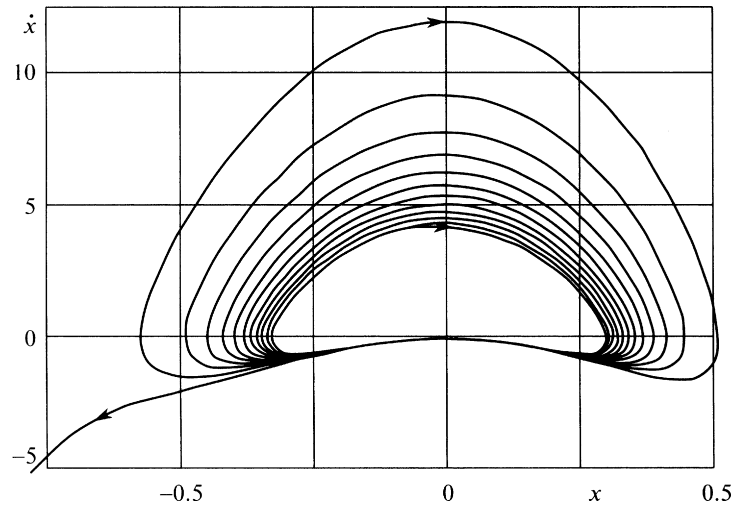

Fig. 10.

On account of the effects of the flattening of the trajectories described here, students in the years from 1950 to 1960 would have been unable, in the case of these parameter values, to reveal limit cycles in system (2.1). In order to calculate the trajectories shown in Figs. 9 and 10, a powerful contemporary computer procedure was required.[39]

Note that inclusion of computer calculations recently enabled a two-dimensional cubic system with twelve cycles to be found.[43]

4. The method of harmonic linearization (MHL)

We first recall some concepts and ideas which are widely known in control theory.

Any continuous linear device L with a scalar input and output and an n-dimensional state space (Fig. 11) is described by the equations

$$\frac{dx}{dt} = Px + qu, \quad \sigma = r^*x$$

where P is a real $n \times n$ matrix, q and r are real n-vectors and an asterisk denotes transposition. Here, the function $u(t)$ is treated as an input (control) and the function $\sigma(t)$ as an output (observation).

If P, q and r are independent of time, L is said to be a time-independent device. Here, we shall consider time-independent devices. In the case of such devices, it is possible to introduce a transfer function [44–47]

$$W(p) = r^*(P - pI)^{-1}q$$

where p is a complex variable.

Linear devices with non-linear feedback (Fig. 12) are one of the basic subjects of investigations in non-linear control theory. Here, the input $\sigma(t)$ and the output $u(t)$ of the device N are connected by the following functional relation

$$u(t) = \psi(\sigma(t))$$

Hence, the block diagram shown in Fig. 12 is described by the system of differential equations

$$\frac{dx}{dt} = Px + q\psi(r^*x) \tag{4.1}$$

We shall subsequently assume that $\psi(0) = 0$ and $\psi(\sigma)$ is a piecewise continuous function (that is, it has a finite number of discontinuities of the first kind in any finite interval.[48] The solutions of system (4.1) are understood in Filippov's sense.[49]

The method of harmonic linearization (MHL), which is also known as the method of describing functions, is a common approximate method (that is, not rigorously mathematically substantiated) of searching for oscillations which are close to the harmonic periodic oscillations of non-linear dynamical systems.[44-47]

We will describe the standard procedure for searching for harmonic oscillations using the MHL in the case of system (4.1).

In searching for such an oscillation, a certain harmonic linearization coefficient k is introduced so that the matrix of the linear system

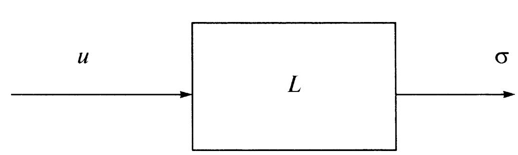

Fig. 11.

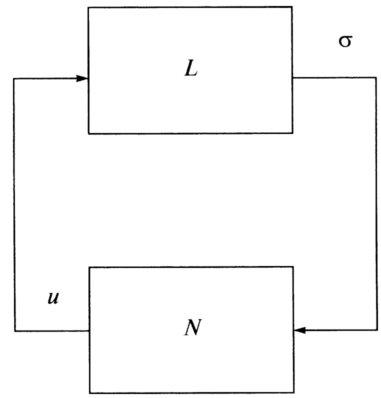

Fig. 12.

$$\frac{dx}{dt} = P_0 x \tag{4.2}$$

where $P_0 = P + kqr^*$, has a pair of pure imaginary eigenvalues $\pm i\omega_0(\omega_0 > 0)$ and its remaining eigenvalues have negative real parts. We will assume here that such a number k exists.

A transfer function $W(p)$ is used to solve practical problems of determining the quantities k and ω_0: the number ω_0 is determined from the equation

$$\operatorname{Im} W(i\omega_0) = 0$$

and k is then calculated using the formula

$$k = -(\operatorname{Re} W(i\omega_0))^{-1}$$

If such ω_0 and k are found, then it is confirmed that system (4.1) has a periodic solution $x(t)$ for which

$$\sigma(t) = r^* x(t) \approx a\cos\omega_0 t \tag{4.3}$$

At the same time, the amplitude a is found from the equation

$$\int_0^{2\pi} \psi(a\cos\tau)\cos\tau d\tau = ak\pi \tag{4.4}$$

It is well known that the MHL can produce incorrect results. Such results have been presented in the case of bang-bane systems.[50] We recall the well known Aizerman problem for smooth non-linear systems.[9]

In 1949, Aizerman formulated the following problem for system (4.1): suppose all linear systems (4.1) with $\psi(\sigma) = \mu\sigma$, $\mu \in (\mu_1, \mu_2)$ are asymptotically stable. It is necessary to ascertain whether any non-linear system (4.1) with non-linearity $\psi(\sigma)$ satisfying the condition

$$\mu_1 < \psi(\sigma)/\sigma < \mu_2, \quad \forall\sigma \neq 0$$

will be stable as a whole (that is, the zero solution is asymptotically stable and any solution $x(t)$ tends to zero when $t \to +\infty$).

We will now consider the result that the MHL gives for this problem.

It is clear that, in this case, one of the conditions: either $k > \mu_2$ or $k < \mu_1$ is satisfied for the harmonic linearization coefficient k. But, then, either $k\sigma^2 > \psi(\sigma)\sigma$ or $k\sigma^2 < \psi(\sigma)\sigma$ for any values of σ. It follows from this that the inequality

$$\int_0^{2\pi/\omega_0} (\psi(a\cos\omega_0 t)\cos\omega_0 t - ka(\cos\omega_0 t)^2)dt \neq 0$$

holds for all $a \neq 0$. Comparing this inequality with relation (4.4), we obtain that the non-linear system considered does not have a periodic solution for which relation (4.3) is satisfied. Hence, within the limits of the application of the MHL, the conclusion is usually drawn that the non-linear system (4.1) does not have any periodic solutions at all and is stable as a whole [51].

It will be shown later that these conclusions are incorrect and that the non-linear systems considered in the Aizerman problem can have periodic solutions.

Over the course of many decades attempts have been made, in relation to the above-mentioned facts, to find the classes of systems where MHL (or the various generalizations of it) turns out to be accurate and gives correct results. Some of the first attempts in this direction were Bulgakov's papers,[52,53] where a version of the classical small parameter method was used. This was subsequently subjected to serious criticism [54] based on the fact that "these small parameter methods rest on the assumption that the initial system differs only slightly from a linear system possessing a natural generating frequency. Such assumptions cannot be made in automatic control theory since the system is clearly non-conservative and the stability conditions in the linear approximation are satisfied with a sufficient margin".[54]

Other methods of introducing a small parameter, based on a filter hypothesis, started to be developed which took account of this criticism.[44,45,55-59]

The development of numerical methods, computational techniques and the applied theory of bifurcations enabled a return to be made to earlier ideas on the application of the small-parameter method and MHL in dynamical system and enabled them to be considered from new positions.

The use of modern computer calculations makes it possible to find without difficulty an asymptotically stable periodic solution of a system of ordinary differential equations using initial data found in the region of attraction of this solution. In other words, from an initial point located in the region of attraction of the required solution after a transitional process, the computational procedure reaches a periodic solution and calculates it.

On the other hand, it has been shown above that very significant difficulties are encountered in the numerical search for periodic solutions in the whole of phase space even in the case of two-dimensional autonomous quadratic systems.

The combined use of MHL, the classical small parameter method and numerical methods enables one to reduce the calculation of the periodic modes to a certain multistep procedure, where MHL is used in the first step.

We will now describe the basic stages of this procedure for system (4.1).

As a result of introducing a harmonic linearization coefficient, system (4.1) can be rewritten in the form

$$\frac{dx}{dt} = P_0x + q\varphi(r^*x); \quad \varphi(\sigma) = \psi(\sigma) - k\sigma \tag{4.5}$$

Since it is the periodic solutions of system (4.5) which are of interest, it is quite natural to introduce a finite sequence of piecewise continuous functions $\varphi_0(\sigma), \varphi_1(\sigma), \ldots, \varphi_m(\omega)$ such that the function $\varphi_0(\sigma)$ is small, $\varphi_m(\sigma) = \varphi(\sigma)$ and the graphs of the neighbouring functions φ_j and φ_{j+1} are slightly different in a certain sense. In the case of continuous functions, this is, for example, the differences between the functions at each point (for piecewise continuous functions, it is the same smallness but only outside small neighbourhoods of the points of discontinuity).

The smallness of the function $\varphi_0(\sigma)$ enables one to use and substantiate the MHL in this case for the system

$$\frac{dx}{dt} = P_0x + q\varphi_0(r^*x) \tag{4.6}$$

by determining the solution close to the stable harmonic periodic solution $x(t) = x_0(t)$. All points of this stable periodic solution are either located in the region of attraction of the stable periodic solution $x(t) = x_1(t)$ of the system

$$\frac{dx}{dt} = P_0x + q\varphi_j(r^*x) \tag{4.7}$$

with $j=1$ or, on changing from system (4.6) to system (4.7) with $j=1$, stability loss bifurcation and the disappearance of the periodic solution is observed. In the first case, $x_1(t)$ can be determined numerically, producing the trajectory of system (4.7) with $j=1$ from the initial point $x_0(0)$.

Starting from the point $x_0(0)$, the computational procedure, after the transition process, reaches the periodic solution $x_1(t)$ and calculates it. To do this, the interval $(0, T)$, in which the calculation is carried out, must be sufficiently wide.

After calculating $x_1(t)$, it is possible to change to the following system (4.7) with $j=2$ and to organize a similar procedure for calculating the periodic solution $x_2(t)$, producing a trajectory from the initial point $x(0) = x_1(T)$ which, as t increases, approaches the periodic trajectory $x_2(t)$.

Continuing this procedure further and calculating $x_j(t)$ using the trajectories of system (4.7) with the initial data $x_j(0) = x_{j-1}(T)$, we either arrive at the calculation of the periodic solution of system (4.7) with $j=m$, that is, of the initial system (4.5), or we observe stability loss bifurcation and the disappearance of the periodic solution at a certain step.

The functions

$$\varphi_0(\sigma) = \varepsilon\varphi(\sigma), \quad \varphi_1(\sigma) = \varepsilon_1\varphi(\sigma), \ldots, \varphi_{m-1}(\sigma) = \varepsilon_{m-1}\varphi(\sigma), \quad \varphi_m(\sigma) = \varphi(\sigma)$$

where ε is a "classical" small positive parameter and, for example,

$$\varepsilon_j = j/m, \quad j = 1, \ldots, m$$

is the simplest and most natural class of functions φ_j in the procedure described above.

A rigorous substantiation of the MHL and the determination of the initial conditions for which system (4.6) has a stable periodic solution, close to a harmonic solution, is found to be possible in the case of system (4.6) with the function $\varphi_0(\sigma)$.

Since, in the procedure described here, the small parameter method together with the MHL only determines the "starting" initial conditions for the "starting" periodic solution, all estimates for this determination can be considerably simplified and roughly approximated. This especially concerns the rigorous results on the stability of a periodic solution. Here, the stability of the solution can be replaced by the stability of a certain Poincaré mapping.

We will next describe such a simplified analysis of systems (4.6) with a small parameter ε.

5. Estimation of the solutions of a system containing a small parameter

An estimate of the increase in the solution of system (4.6) and an estimate of the difference between the solutions of systems (4.6) and (4.2) in a finite time interval are found to be necessary in order to compare the solutions of systems (4.2) and (4.6) with the same initial data. We now present these simple estimates.

We shall assume that the estimate

$$|\varphi_0(\sigma)| \le N(R)\varepsilon, \quad \forall\sigma \in [-R, R] \tag{5.1}$$

is satisfied for the function $\varphi_0(\sigma)$ and consider the solutions $x(t)$ and $z(t)$ of system (4.6) and (4.2) with the same initial data $x(0) = z(0)$.

Lemma 1. *Suppose the inequality*

$$|z(t)| \le \rho, \quad \forall t \in [0, T] \tag{5.2}$$

is satisfied. Then, for a number $\varepsilon > 0$, which is sufficiently small with respect to T, ρ, $|P_0|$, $|SP_0|^{-1}$, $N(2\rho|r|)$, the solution $x(t)$ satisfies the estimates

$$|x(t)| \le 2\rho, \quad \forall t \in [0, T] \tag{5.3}$$

Proof. We assume that estimate

$$|x(t) - z(t)| \le \varepsilon N(2\rho|r|)|q||P_0|^{-1}\exp(|P_0|T), \quad \forall t \in [0, T] \tag{5.4}$$

Proof. We assume that estimate (5.3) is not satisfied. The existence of a number $\tau \in (0, T)$ then follows from the continuity of $x(t)$ and the inequality $|x(0)| \le \rho$ for which

$$|x(t)| \le 2\rho, \quad \forall t \in [0, \tau]; \quad |x(\tau)| = 2\rho \tag{5.5}$$

We now introduce the function $V(t|x(t) - z(t)|$ into the treatment. It is continuous in $[0, \tau]$ and, therefore, $V(t) \ne 0$ in intervals of the form $(\alpha, \beta) \in [0, \tau]$. The equality $V(t) = 0$ is satisfied at the remaining points of the interval $[0, \tau]$.

We now construct estimates in one of the intervals (α, β). The inequality

$$(V(t)^2)^{\bullet} \le 2|P|V(t)^2 + 2|q|N(2|r|\rho)\varepsilon V(t), \quad \forall t \in (\alpha, \beta)$$

which can be rewritten in the form

$$(V(t) + \varepsilon N(2|r|\rho)|q||P|^{-1})^{\bullet} \le |P|(V(t) + \varepsilon N(2|r|\rho)|q||P|^{-1}), \quad \forall t \in (\alpha, \beta)$$

follows from relation (5.1).

From this differential inequality and from the assumption that the interval (α, β) to the left is a maximum interval (that is $V(\alpha) = 0$), we obtain the estimate

$$V(t) \le \varepsilon N(2|r|\rho)|q||P|^{-1}\exp(|P|T), \quad \forall t \in (\alpha, \beta) \tag{5.6}$$

Since the intervals (α, β) are all possible intervals for which $V(t) \ne 0$, we conclude that the estimate (5.6) holds for all $t \in [0, \tau]$.

Choosing ε such that the inequality

$$\varepsilon N(2|r|\rho)|q||P|^{-1}\exp(|P|T) < \rho$$

is satisfied, from inequality (5.6) we obtain the estimate

$$|x(t)| \le |x(t) - z(t)| + |z(t)| < 2\rho, \quad \forall t \in [0, \tau]$$

which contradicts the last relation of (5.5). The resulting contradiction proves estimate (5.3).

It follows from estimate (5.3) that inequality (5.6) is satisfied when $t \in [0, T]$. This implies that inequality (5.4) is satisfied which also proves Lemma 1.

It follows from Lemma 1 that the solutions of systems (4.2) and (4.6) differ by no more than $O(\varepsilon)$.

6. Poincaré mapping for harmonic linearization in the basic case

Here, we now consider system (4.6) with $\varphi_0(\sigma) = \varepsilon\varphi(\sigma)$, where ε is a small parameter and $\varphi(\sigma)$ is a piecewise continuous function with points of discontinuity ν_j and $\varphi(0) = 0$. We shall assume that the function $\varphi(\sigma)$ satisfies the Lipschitz conditions

$$|\varphi(\sigma_1) - \varphi(\sigma_2)| \le L(\nu_j, \nu_{j+1})|\sigma_1 - \sigma_2|, \ \forall\sigma_1 \in (\nu_j, \nu_{j+1}), \ \forall\sigma_2 \in (\nu_j, \nu_{j+1})$$

in the intervals (ν_j, ν_{j+1}).

If the function does not have points of discontinuity in $(\nu_j, +\infty)$ (or $(-\infty, \nu_j)$), we shall assume that

$$|\varphi(\sigma_1) - \varphi(\sigma_2)| \le L(\nu_j, R)|\sigma_1 - \sigma_2|, \ \forall\sigma_1 \in (\nu_j, R), \ \forall\sigma_2 \in (\nu_j, R); \ R \in (\nu_j, +\infty)$$

In the case of $(-\infty, \nu_j)$, we have the intervals (R, ν_j), where $R \in (-\infty, \nu_j)$.

In the case when there are no points of discontinuity, we assume that

$$|\varphi(\sigma_1) - \varphi(\sigma_2)| \le L(R)|\sigma_1 - \sigma_2|, \quad \forall\sigma_1 \in (-R, R), \quad \forall\sigma_2 \in (-R, R)$$

where R is an arbitrary positive number.

We now introduce the notation

$$N(R) = \max_{\sigma \in [-R, R]} |\varphi(\sigma)|$$

It is well known that system (4.6) can be reduced by a non-singular linear transformation to the form

$$\begin{aligned} \dot{x}_1 &= -\omega_0 x_2 + b_1 \varepsilon\varphi(x_1 + c^* x_3) \\ \dot{x}_2 &= \omega_0 x_1 + b_2 \varepsilon\varphi(x_1 + c^* x_3) \\ \dot{x}_3 &= A x_3 + b \varepsilon\varphi(x_1 + c^* x_3) \end{aligned} \tag{6.1}$$

Here A is a constant $(n\text{-}2) \times (n - 2)$ matrix, all the eigenvalues of which have negative real parts, b and c are $(n - 2)$-dimensional vectors and b_1 and b_2 are certain numbers.

Without loss of generality, it can be assumed here that a positive number α exists for the matrix A, and for this number,

$$x_3^*(A + A^*)x_3 \le -2\alpha|x_3|^2, \quad \forall x_3 \in R^{n-2} \tag{6.2}$$

It is obvious that system (4.2) reduces to the form

$$\dot{z}_1 = -\omega_0 z_2, \quad \dot{z}_2 = \omega_0 z_1, \quad \dot{z}_3 = A z_3 \tag{6.3}$$

We now introduce the following set in the phase spaces of systems (6.1) and (6.3):

$$\Omega = \{|x_3| \le D\varepsilon, x_2 = 0, \ x_1 \in [a_1, a_2]\}$$

Here, a_1, a_2, D are certain positive numbers which will be determined below.

We first introduce the Poincaré mapping F_0 of the set Ω for the trajectories of system (6.3). Here,

$$z_1(t) = \cos(\omega_0 t) z_1(0), \quad z_2(t) = \sin(\omega_0 t) z_1(0), \quad z_3(t) = \exp(At) z_3(0) \tag{6.4}$$

Hence, the first time of intersection of the set $\{z_2 = 0, z_1 \in [a_1, a_2]\}$ by the trajectory of system (6.3), which has been emitted when $t = 0$ from the set Ω, is equal to $2\pi/\omega_0$. Here,

$$F_0 \begin{Vmatrix} z_1 \\ 0 \\ z_3 \end{Vmatrix} = \begin{Vmatrix} z_1 \\ 0 \\ \exp(2A\pi/\omega_0) z_3 \end{Vmatrix}$$

It follows from condition (6.2) that

$$|z_3(2\pi/\omega_0)| < |z_3(0)|$$

Hence, F_0 maps the set Ω into itself.

From formulae (6.4) using Lemma 1 (estimate (5.4)), we obtain the relations for the solutions of system (6.1)

$$\begin{aligned} &x_1(t) = \cos(\omega_0 t) z_1(0) + O(\varepsilon), \quad x_2(t) = \sin(\omega_0 t) z_1(0) + O(\varepsilon), \\ &x_3(t) = \exp(At) z_3(0) + O(\varepsilon) \end{aligned} \tag{6.5}$$

We recall that, here,

$$x_1(0) = z_1(0), \quad x_2(0) = z_2(0) = 0, \quad x_3(0) = z_3(0) \tag{6.6}$$

It follows from formulae (6.5) that, for any point $x_0(0)$, $x_2(0) = 0$, $x_3(0)$ belonging to Ω, a number

$$T = T(x_1(0), x_3(0)) = 2\pi/\omega_0 + O(\varepsilon)$$

exists for which

$$x_1(T) > 0, \quad x_2(T) = 0 \tag{6.7}$$

At the same time, the relations

$$x_1(t) > 0, \quad x_2(t) = 0$$

are not satisfied when $t \in (0, T)$.

The inequality

$$|x_3(t)| \le e^{-\alpha t}\left(|x_3(0)| + \varepsilon \int_0^t e^{\alpha\tau}|b||\varphi(x_1(\tau) + c^* x_3(\tau))| d\tau\right) \tag{6.8}$$

follows from the last equation of system (6.1) and condition (6.2).

If $|x_3(0)| \in D\varepsilon$, it follows from the first two formulae of (6.5) that, for a certain $D_1 > 0$.

$$|x_3(t)| \le D_1\varepsilon, \quad \forall t \in [0, T] \tag{6.9}$$

We next assume that ε is so small that $\varepsilon|c|D_1 < 1$ and, in the first two relations of (6.5), $O(\varepsilon) < 1$. Then,

$$|x_1(t) + c^*x_3(t)| \le 2 + |x_1(0)|, \quad \forall t \in [0, T] \tag{6.10}$$

It follows from inequalities (6.8) and (6.10) that

$$|x_3(t)| \le \varepsilon\left(De^{-\alpha t} + \frac{|b|N(2 + |x_1(0)|)}{\alpha}\right) \tag{6.11}$$

Relations (6.11) and the inequality

$$D\exp\left(-\alpha\frac{2\pi}{\omega_0}\right) + 2\frac{|b|N(2 + |x_1(0)|)}{\alpha} < D \tag{6.12}$$

imply the estimate

$$|x_3(T)| \le D\varepsilon \tag{6.13}$$

We now consider the function

$$V(x_1, x_2) = x_1^2 + x_2^2$$

The relation

$$\dot{V} = 2\varepsilon\varphi(x_1(t) + c^*x_3(t))(b_1x_1(t) + b_2x_2(t)) \tag{6.14}$$

is satisfied along the trajectories of system (6.1) for the derivative of the function V.

We shall now consider the values of $x_1(0)$, where $x_1(0) \neq \pm\nu_j$ and ν_j are the points of discontinuity of the function $\varphi(\sigma)$.

From the first two relations of (6.5) and inequality (6.9), we obtain the estimate

$$\begin{aligned} &V(x_1(T), x_2(T)) - V(x_1(0), x_2(0)) \\ &= 2\varepsilon\int_0^T \varphi(\cos(\omega_0 t)x_1(0))(b_1\cos(\omega_0 t)x_1(0))dt + O(\varepsilon^2) \end{aligned} \tag{6.15}$$

Actually, in the case of values of t for which $\cos(\omega_0 t)x_1(0) \neq \nu_j$, from the Lipschitz property of the function $\nu(\sigma)$, the first two relations of (6.5) and inequality (6.9) in the case of small ε we obtain the estimate

$$\varphi(x_1(t) + c^*x_3(t)) = \varphi(\cos(\omega_0 t)x_1(0)) + O(\varepsilon) \tag{6.16}$$

If $\cos(\omega_0 t)x_1(0) = \nu_j$, then, from the inequality $x_1(0) \neq \pm\nu_j$, we obtain that $\omega_0 t \neq k\pi$ and, consequently, in the case of small ε,

$$|\dot{x}(t) + c^*\dot{x}_3(t)| = |\omega_0\cos(\omega_0 t)x_1(0)| + O(\varepsilon) > \kappa > 0$$

where κ is a certain number which is independent of ε.

It follows from the last inequality that estimate (6.16) is satisfied for all values of $t \in [0, T]$ for which $\omega_0 t \neq k\pi$. Hence, relation (6.15) follows from the first two relations of (6.5) and equality (6.14).

Since $T = 2\pi/\omega_0 + O(\varepsilon)$, from equality (6.15) we obtain the estimate

$$x_1(T)^2 - x_1(0)^2 = 2\varepsilon b_1\int_0^{2\pi/\omega_0} \varphi(\cos(\omega_0 t)x_1(0))(\cos(\omega_0 t)x_1(0))dt + O(\varepsilon^2) \tag{6.17}$$

We now introduce the function

$$K(a) = \int_0^{2\pi/\omega_0} \varphi(\cos(\omega_0 t)a)\cos(\omega_0 t)dt$$

The following result arises from relations (6.13) and (6.17).

Theorem 4. *If the inequalities $a_1 \neq \pm\nu_j$, $a_2 \neq \pm\nu_j$ and*

$$b_1K(a_1) > 0, \quad b_1K(a_2) < 0 \tag{6.18}$$

are satisfied, then, for a sufficient small $\varepsilon > 0$, the Poincaré mapping

$$F\begin{Vmatrix} x_1(0) \\ 0 \\ x_3(0) \end{Vmatrix} = \begin{Vmatrix} x_1(T) \\ 0 \\ x_3(T) \end{Vmatrix} \tag{6.19}$$

of the set Ω is a mapping into itself: $F\Omega \subset \Omega$.

The following corollary follows from this theorem and Brouwer's fixed point theorem.[60]

Corollary 1. *If the inequalities $a_1 \neq \pm\nu_j$, $a_2 \neq \pm\nu_j$ and (6.18) are satisfied, then, for sufficiently small $\varepsilon > 0$, system (6.1) has a periodic solution with period*

$$T = \frac{2\pi}{\omega_0} + O(\varepsilon)$$

This solution is stable in the sense that its neighbourhood of Ω is mapped into itself: $F\Omega \subset \Omega$.

The following result is proved in a similar manner.

Theorem 5. *If the inequalities $a_1 \neq \pm\nu_j$, $a_2 \neq \pm\nu_j$ and inequalities (6.18) of the opposite sign are satisfied, then, for sufficiently small $\varepsilon > 0$, the Poincaré mapping (6.19) of the set Ω is of a hyperbolic character: compression occurs along x_3, estimate (6.13) is satisfied and elongation occurs along x_1: $Fa_1 < a_1$, $Fa_2 > a_2$.*

7. Algorithm for determining the stable periodic solutions of generating systems

We will write the transfer function of system (4.6) from the "input ν_0" to the "output $-r^*x$"

$$W(p) = r^*(P_0 - pI)^{-1}q = \frac{\alpha p + \beta}{p^2 + \omega_0^2} + \frac{R(p)}{Q(p)} \tag{7.1}$$

where α and β are certain numbers, $Q(p)$ is a stable n - 2 degree polynomial and $R(p)$ is a polynomial of degree less than n - 2. We shall assume that $R(p)$ and $Q(p)$ do not have common roots.

Since system (6.1) is equivalent to (4.6) and, consequently, the transfer functions of these systems are identical, we have the relations

$$\alpha = -b_1, \quad \beta = b_2\omega_0, \quad \frac{R(p)}{Q(p)} = c^*(A - pI)^{-1}b$$

Theorems 4 and 5 can be reformulated as follows.

Theorem 6. *If the conditions*

$$K(a) = 0, \quad \alpha\frac{dK(a)}{da} > 0, \quad a \neq \pm\nu_j$$

are satisfied, then, in the case of sufficiently small $\varepsilon > 0$, system (4.6) with the transfer function (7.1) has a T-periodic solution such that

$$r^*x(t) = a\cos(\omega_0 t) + O(\varepsilon), \quad T = 2\pi/\omega_0 + O(\varepsilon)$$

This periodic solution is stable in the sense that some ε-neighbourhood of it exists such that all solutions with initial data from this ε-neighbourhood remains in it when the time t increases.

Theorem 7. *If the conditions*

$$K(a) = 0, \quad \alpha\frac{dK(a)}{da} < 0, \quad a \neq \nu_j$$

are satisfied, then, for sufficiently small $\varepsilon > 0$, system (4.6) with transfer function (7.1) has a solution of the form

$$r^*x(t) = a\cos(\omega_0 t) + O(\varepsilon), \quad \forall t \in [0, 2\pi/\omega_0]$$

and the behaviour of the trajectories in the neighbourhood of this solution has a hyperbolic character.

Theorems 6 and 7 are identical to the procedure for searching for the stable and unstable periodic solutions by the standard approximate MHL[43-46] in the case of small ε.

Example 1. *Suppose $\varphi(\sigma) = \sigma - \operatorname{sign}\sigma$. Then,*

$$K(a) = (\pi a - 4)/\omega_0$$

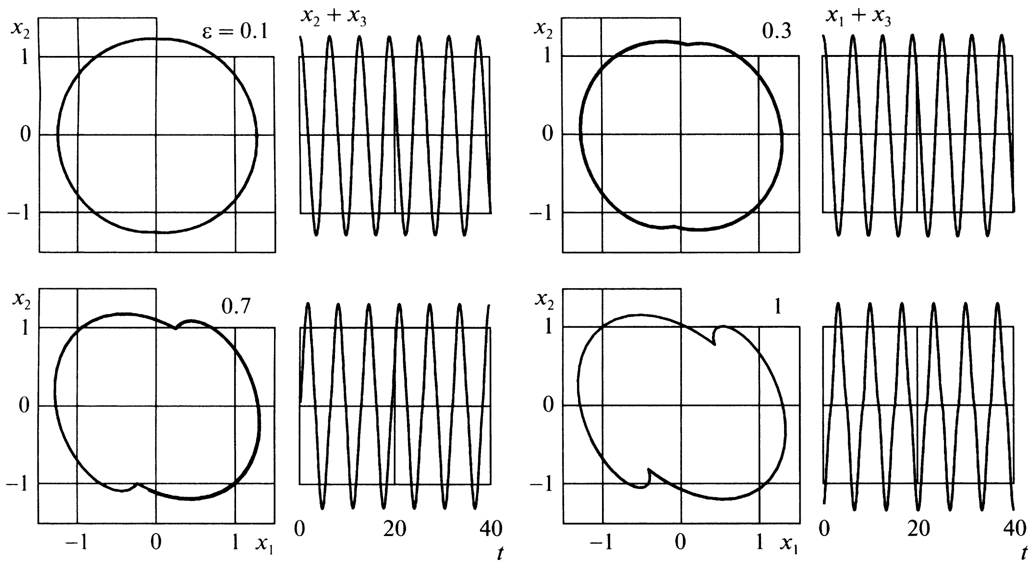

Fig. 13.

It follows from this that a is determined from the equations $K(a) = 0$ in the following manner:

$$a = a_0 = 4/\pi$$

and the stability conditions takes the form $\alpha > 0$.

We now consider system (4.5) with

$$W(p) = \frac{p+1}{p^2+1} - \frac{1}{p+1} \tag{7.2}$$

and then systems (4.6) and (6.1). In this case,

$$\omega_0 = 1, \quad b_1 = -1, \quad b_2 = 1, \quad A = -1, \quad b = 1, \quad c = 1$$

Using the classical MHL,[43-46] we obtain that, for any $\varepsilon > 0$, system (4.6) (or (6.1)) has a periodic solution and for it

$$\sigma(t) = r^* x(t) \approx a_0 \cos t$$

According to Theorem 6 in the case of small $\varepsilon > 0$, system (6.1) considered here has a stable periodic solution of the form

$$x_1(t) = a_0 \cos t + O(\varepsilon), \quad x_2(t) = a_0 \sin t + O(\varepsilon), \quad x_3(t) = O(\varepsilon)$$

We next trace the transformation of this solution as ε is increased in discrete steps from 0.1 to 1. The computational procedures are organized as described in Section 4.

The projections of the periodic solutions calculated in this way onto the $\{x_1, x_2\}$ plane are shown in Fig. 13. The graph of $\sigma(t) = x_1(t) + x_3(t)$ is also shown for them.

Note that the "output" $\sigma(t)$ is close to harmonic and the filter hypothesis [54-59] holds here. Hence, it is possible in principle in this case to substantiate the standard MHL for the values of ε considered.

Example 2. *Now suppose $\varphi(\sigma) = k_1\sigma + k_3\sigma^3$. Then,*

$$K(a) = \left(k_1 a + \frac{3}{4} k_3 a^3\right) \frac{\pi}{\omega_0}$$

It follows from this that a is determined from the equation $K(a) = 0$ as follows

$$a = a_1 = \sqrt{-\frac{4k_1}{3k_3}}$$

and the stability condition takes the form $\alpha k_1 < 0$.

We again consider system (4.5) with $W(p)$ of the form of (7.2). Suppose $k_1 = -3$, $k_3 = 4$. Then $a_1 = 1$.
Using the standard MHL, we obtain that, for any $\varepsilon > 0$, system (4.6) (or (6.1)) has a periodic solution and for it

$$\sigma(t) = r^* x(t) \approx \cos t$$

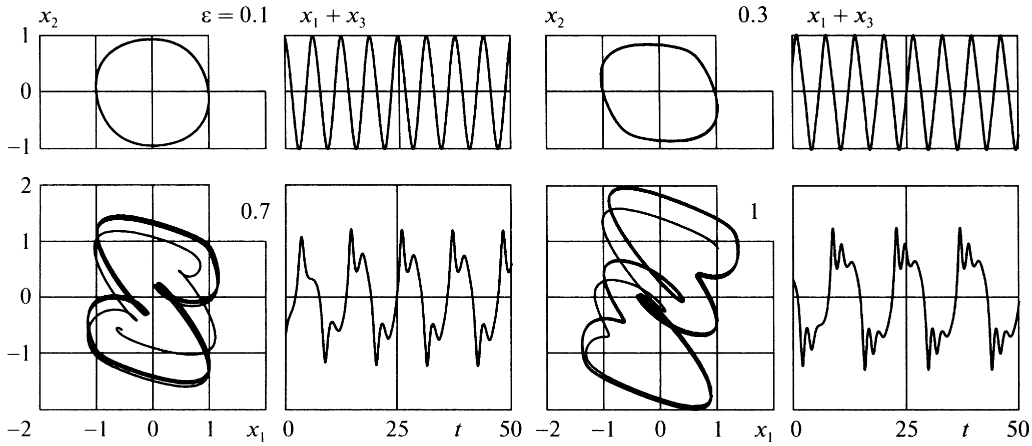

Fig. 14.

According to Theorem 6 in the case of small $\varepsilon > 0$, system (6.1) here has a periodic solution of the form

$$x_1(t) = \cos t + O(\varepsilon), \quad x_2(t) = \sin t + O(\varepsilon), \quad x_3(t) = O(\varepsilon) \tag{7.3}$$

Next, using the computational procedure described in Section 4, we obtain the periodic solution of system (4.7) when

$$\varphi_j(\sigma) = \varepsilon_j \varphi(\sigma), \quad \varepsilon_1 = 0.1, \quad \varepsilon_2 = 0.3, \quad \varepsilon_3 = 0.7, \quad \varepsilon_4 = 1$$

The projections of the periodic solutions calculated in this manner into the $\{x_1, x_2\}$ plane are shown in Fig. 14. The graph of $\sigma(t) = x_1(t) + x_3(t)$ is also shown for these periodic solutions. When $\varepsilon_3 = 0.7$ and $\varepsilon_4 = 1$, the output $\sigma(t)$ is far from harmonic and, in this case, the filter hypothesis is incorrect. Here, it is therefore impossible in principle to substantiate the MHL on the basis of the filter hypothesis.

We will now present an example of collapses of periodic solutions when the parameter ε_j in the computational procedures described in Section 4 is increased.

Example 3. *Suppose $\varphi(\sigma) = -3\sigma + 4\sigma^3$ and*

$$W(p) = \frac{p+1}{p^2+1} + \frac{1}{p+1}$$

In this case,

$$\omega_0 = 1, \quad b_1 = -1, \quad b_2 = 1, \quad A = -1, \quad b = -1, \quad c = 1$$

The standard MHL predicts the existence of a periodic solution for any $\varepsilon > 0$ and, for it, $\sigma(t) = x_1(t) + x_3(t) \approx \text{const}$.

According to Theorem 4, in the case of a small $\varepsilon > 0$ we obtain a periodic solution of the form (7.3).

As ε increases, a periodic solution of system (6.1) exists when $\varepsilon \in (0, \rho)$, and, when $\varepsilon = \rho$, a stability loss bifurcation occurs and the periodic solution disappears. For $\varepsilon \in (\rho, 1)$, the domain where a periodic trajectory is found is the domain of attraction of the stable equilibrium state. Projections of the periodic solutions onto the $\{x_1, x_2\}$ plane are shown in Fig. 15 for $\varepsilon = 0.25, 0.3, 0.35$.

When $\varepsilon = 0.35$, a tendency towards the equilibrium state of the trajectories with initial data on the "collapsed periodic trajectory" is observed.

We recall that the equilibrium states of system (4.6) satisfy the relations

$$\sigma_0 + W(0)\varepsilon\varphi(\sigma_0) = 0, \quad x_0 = -(P + kqr^*)^{-1} q\varepsilon\varphi(\sigma_0)$$

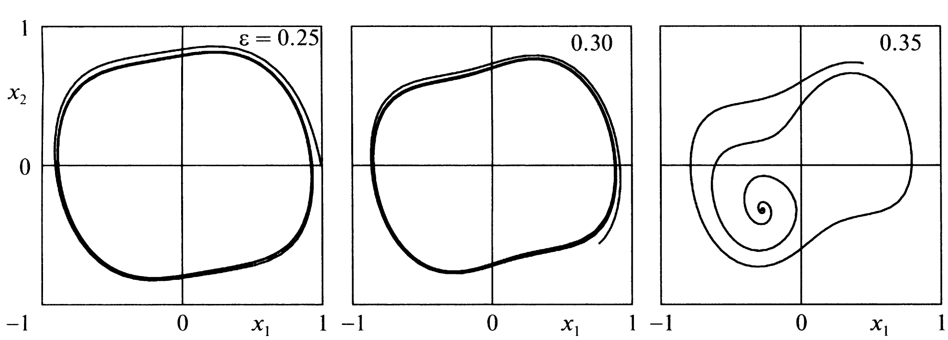

Fig. 15.

Since, here, $W(0) = 2$, from the first equality we obtain the relation

$$\sigma_0(1 - 6\varepsilon + 8\varepsilon\sigma_0^2) = 0$$

It follows from this that, when $\varepsilon < 1/6$, the system considered only has a zero equilibrium. When $\varepsilon > 1/6$, the system has three equilibrium states

$$\sigma_0 = 0, \quad \sigma_0 = \pm\sqrt{(6\varepsilon - 1)/(8\varepsilon)}$$

They are unstable when $\varphi'(\sigma_0) < 0$ and stable when $\varphi'(\sigma_0) > 0$. Consequently, the zero equilibrium is always unstable, and the non-zero equilibria are unstable when $\varepsilon < 1/4$ and stable when $\varepsilon > 1/4$. The periodic solution considered, which collapsed after bifurcation, is also attracted to one of these equilibria. The standard MHL "does not notice" all these qualitative changes in the phase space of the system considered.

8. Poincaré mapping for harmonic linearization in the critical case

It has been shown in Section 4 that it is necessary to develop more refined "non-standard" MHL in order to find the periodic solution in systems satisfying the generalized Routh–Hurwitz conditions. This extension of the standard MHL to several singular mathematical constructions and estimates in the spirit of classical investigations of critical cases in the theory of the stability of motion [61] enabled effective estimates to be obtained for periodic oscillations in non-linear systems satisfying the generalized Routh–Hurwitz conditions.

Here, we consider the development of this method for systems (4.6) with non-linearities $\varphi_0(\sigma)$ of the special form

$$\varphi_0(\sigma) = \begin{cases} \mu\sigma, & \forall\sigma \in (-\varepsilon, \varepsilon) \\ M\varepsilon^3, & \forall\sigma > \varepsilon \\ -M\varepsilon^3, & \forall\sigma < -\varepsilon \end{cases} \tag{8.1}$$

where μ and M are certain positive numbers and ε is a small positive parameter. Non-linearities of close types have been used in theorems on the existence of periodic solutions for systems satisfying the generalized Routh-Hurwitz conditions.[13,62–64]

We reduce system (4.6) with non-linearity (8.1) to the form

$$\begin{aligned} \dot{x}_1 &= -\omega_0 x_2 + b_1\varphi_0(x_1 + c^*x_3) \\ \dot{x}_2 &= \omega_0 x_1 + b_2\varphi_0(x_1 + c^*x_3) \\ \dot{x}_3 &= Ax_3 + b\varphi_0(x_1 + c^*x_3) \end{aligned} \tag{8.2}$$

As in system (6.1), A is a constant $(n - 2) \times (n - 2)$ matrix, all the eigenvalues of which have negative real parts, b and c are constant $(n-2)$-vectors and b_1 and b_2 are certain numbers.

Without loss of generality, we shall assume that condition (6.2) is satisfied for the matrix A.

The Poincaré mapping will be introduced here using a somewhat different method than in Section 6. The main difference lies in the introduction of the set Ω:

$$\Omega = \{|x_3| \le D\varepsilon^2, x_1 + c^*x_3 = 0, x_2 \in [-a_1, a_2]\}$$

where a_1, a_2 and D are certain positive numbers which will be determined below.

It follows from Lemma 1 that the relations

$$\begin{aligned} x_1(t) &= -\sin(\omega_0 t)x_2(0) + O(\varepsilon) \\ x_2(t) &= \cos(\omega_0 t)x_2(0) + O(\varepsilon) \\ x_3(t) &= O(\varepsilon) \end{aligned} \tag{8.3}$$

hold for all $t \in [0, 4\pi/\omega_0]$ in the case of the solutions of system (8.2) with initial data $x_1(0)$, $x_2(0)$, $x_3(0)$ from the set Ω. It follows from this that, in the case of these solutions, numbers

$$0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < T$$

exist for which

$$\sigma(t) = x_1(t) + c^*x_3(t) \in (0, \varepsilon), \ \forall t \in (0, \tau_1); \ \sigma(\tau_1) = \varepsilon, \ \sigma(t) > \varepsilon, \ \forall t \in (\tau_1, \tau_2)$$

$$\sigma(\tau_2) = \varepsilon, \ \sigma(t) \in (-\varepsilon, \varepsilon), \ \forall t \in (\tau_2, \tau_3); \ \sigma(\tau_3) = -\varepsilon, \ \sigma(t) < -\varepsilon, \ \forall t \in (\tau_3, \tau_4)$$

$$\sigma(\tau_4) = -\varepsilon, \ \sigma(t) \in (-\varepsilon, 0), \ \forall t \in (\tau_4, T); \ \sigma(T) = 0$$

At the same time,

$$\tau_1 = \frac{\varepsilon}{\omega_0|x_2(0)|} + O(\varepsilon^2),\ \ \tau_2 - \tau_1 = \frac{\pi}{\omega_0} + O(\varepsilon),\ \ \tau_3 - \tau_2 = \frac{2\varepsilon}{\omega_0|x_2(0)|} + O(\varepsilon^3)$$
$$\tau_4 - \tau_3 = \frac{\pi}{\omega_0} + O(\varepsilon),\quad T - \tau_4 = \frac{\varepsilon}{\omega_0|x_2(0)|} + O(\varepsilon^2) \tag{8.4}$$

The estimates

$$x_3(t) = e^{At}x_3(0) + \int_0^t e^{A(t-s)}b\varphi_0(x_1(s) + c^*x_3(s))ds$$

successively follow from the formula

$$|x_3(\tau_j)| \le e^{-\lambda\tau_j}|x_3(0)| + O(\varepsilon^2),\quad j = 1, 2, \dots, 5;\quad \tau_5 = T$$

relations (8.3) and (8.4) and the form of the function $\nu_0(\sigma)$. In the inequality corresponding to $j=5$, $O(\varepsilon^2) \le E\varepsilon^2$, where the number E depends on $|b|$, μ and M.

(Note that it follows from inequality (6.2) that $|e^{A(t-s)}| \le 1, \forall t \ge s$). Hence, by choosing the number D such that

$$e^{-\lambda T}D + E < D$$

we obtain that the Poincaré mapping

$$F\begin{Vmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Vmatrix} = \begin{Vmatrix} x_1(T) \\ x_2(T) \\ x_3(T) \end{Vmatrix}$$

of the set Ω satisfies the inclusion

$$F\Omega \subset \{|x_3| \le D\varepsilon^2, x_1 + c^*x_3 = 0, x_2 \in [-a_1 + O(\varepsilon), -a_2 + O(\varepsilon)]\}$$

We shall make this estimate more precise, showing that, for certain values of the parameters, $F\Omega \subset \Omega$. To do this, we introduce the function

$$w(t) = x_1(t)^2 + x_2(t)^2$$

into the treatment, where $x_j(t) (j = 1, 2, 3)$ is the solution of system (8.2) with initial data $x_j(0)$ from Ω.

We now estimate the increment in the function w, representing the integral over the interval $[0,T]$ in the form of a sum of integrals over the intervals $[0, \tau_1]$, $[\tau_1, \tau_2]$, $[\tau_3, \tau_4]$, $[\tau_4, T]$. We obtain

$$w(T) - w(0) = \int_0^T \dot{w}(t)dt = \int_{-\varepsilon}^{\varepsilon}\left(\frac{dw_1}{d\sigma} - \frac{dw_2}{d\sigma}\right)d\sigma - \frac{8M\varepsilon^3 b_1 x_2(0)}{\omega_0} + O(\varepsilon^4);$$
$$w_k(\sigma) = w(t(\sigma))$$

Here, $t(\sigma)$ is a function which is inverse to $\sigma(t)$ in the interval $[0, \tau_1]$ and $[\tau_4, T]$ when $k=1$ and in the interval $[\tau_2, \tau_3]$ when $k=2$.

We now introduce the notation

$$\varkappa(t) = \begin{cases} -1 & \text{When } t \in [0, \tau_1] \cup [\tau_4, T] \\ +1 & \text{When } t \in [\tau_2, \tau_3] \end{cases}$$

The following refinement of the estimates (8.3)

$$x_1(t) = \varkappa(t)\sin(\omega_0 t)x_2(0) + O(\varepsilon^2) = \varkappa(t)\omega_0 t x_2(0) + O(\varepsilon^2)$$
$$x_2(t) = -\varkappa(t)\cos(\omega_0 t)x_2(0) + O(\varepsilon^2) = -\varkappa(t)x_2(0) + O(\varepsilon^2) \tag{8.5}$$

follows from the first two equations of system (8.2) and the estimates (8.4), and the relation

$$\dot{\sigma}(t) = \varkappa(t)\omega_0 x_2(0) + O(\varepsilon)$$

is satisfied.

From this and equalities (8.5), we obtain

$$x_1(t(\sigma)) = \sigma + O(\varepsilon^2), \quad x_2(t(\sigma)) = -\varkappa(t)x_2(0) + O(\varepsilon^2) \tag{8.6}$$

Consequently,

$$\int_{-\varepsilon}^{\varepsilon}\left(\frac{dw_1}{d\sigma} - \frac{dw_2}{d\sigma}\right)d\sigma = 2\int_{-\varepsilon}^{\varepsilon}\left(\frac{b_1\sigma + b_2x_2(0)}{-\omega_0x_2(0) + c^*b\varphi_0(\sigma) + b_1\varphi_0(\sigma)}\right.$$

$$\left.-\frac{b_1\sigma - b_2x_2(0)}{\omega_0x_2(0) + c^*b\varphi_0(\sigma) + b_1\varphi_0(\sigma)} + O(\varepsilon^2)\right)\varphi_0(\sigma)d\sigma$$

$$= 4\int_{-\varepsilon}^{\varepsilon}\frac{(b_2(c^*b + b_1)\varphi_0(\sigma) + b_1\omega_0\sigma)}{\omega_0^2|x_2(0)|}\varphi_0(\sigma)d\sigma + O(\varepsilon^4) = 8\frac{\varepsilon^3}{|x_2(0)|}\chi(\omega_0) + O(\varepsilon^4)$$

$$\chi(\omega_0) = \frac{\mu}{3\omega_0^2}(b_2(c^*b + b_1)\mu + b_1\omega_0)$$

Hence, if the inequality

$$\chi(\omega_0)/a_2 + Mb_1a_2/\omega_0 > 0 \tag{8.7}$$

is satisfied, then, when $x_2(0) = -a_2$, we have $w(T) > w(0)$, and, if the inequality

$$\chi(\omega_0)/a_1 + Mb_1a_1/\omega_0 < 0 \tag{8.8}$$

is satisfied, then, when $x_2(0) = -a_1$, we have $w(T) < w(0)$.

It follows from this that, when inequalities (8.7) and (8.8) are satisfied, the inclusion

$$F\Omega \subset \Omega \tag{8.9}$$

holds, and it follows from this and Brouwer's theorem that a fixed point of the mapping F exists, and this means that a periodic solution of system (8.2) with initial data from the set Ω exists.

It follows from inequalities (87) and (8.8) that these initial data satisfy the relations

$$x_1(0) = O(\varepsilon^2), \quad x_3(0) = O(\varepsilon^2), \quad x_2(0) = -\sqrt{\chi(\omega_0)/M(-b_1)} + O(\varepsilon) \tag{8.10}$$

In this case, the periodic solution is stable in the sense of the inclusion (8.9).

9. Algorithm for determining stable solutions in systems satisfying the generalized Routh–Hurwitz condition

We record the transfer function of system (4.6) (or (8.2)) from the "input φ_0" "to the "output $-r^*x$"

$$W(p) = r^*(P - pI)^{-1}q = \frac{\alpha p + \beta}{p^2 + \omega_0^2} + c^*(A - pI)^{-1}b \tag{9.1}$$

Here,

$$\alpha = -b_1, \quad \beta = b_2\omega_0, \quad c^*b + b_1 = r^*q = -\lim_{p\to\infty} pW(p)$$

We now formulate the results obtained in the preceding section in terms of the transfer function $W(p)$.

Theorem 8. *If the inequality $\alpha > 0$ and*

$$\mu\beta r^*q > \alpha\omega_0^2 \tag{9.2}$$

are satisfied, then system (8.2) with the non-linearity (8.1) has a periodic solution which satisfies relations (8.10) and

$$x_1(0) = O(\varepsilon^2), \quad x_3(0) = O(\varepsilon^2), \quad x_2(0) = -\sqrt{\frac{\mu(\mu\beta r^*q - \alpha\omega_0^2)}{3\omega_0M\alpha}} + O(\varepsilon) \tag{9.3}$$

This solution is stable in the sense of inclusion (8.9).

Example 4. *We consider system (4.6) with the transfer function*

$$W(p) = \frac{p-1}{p^2+1} + \frac{1}{p+1}$$

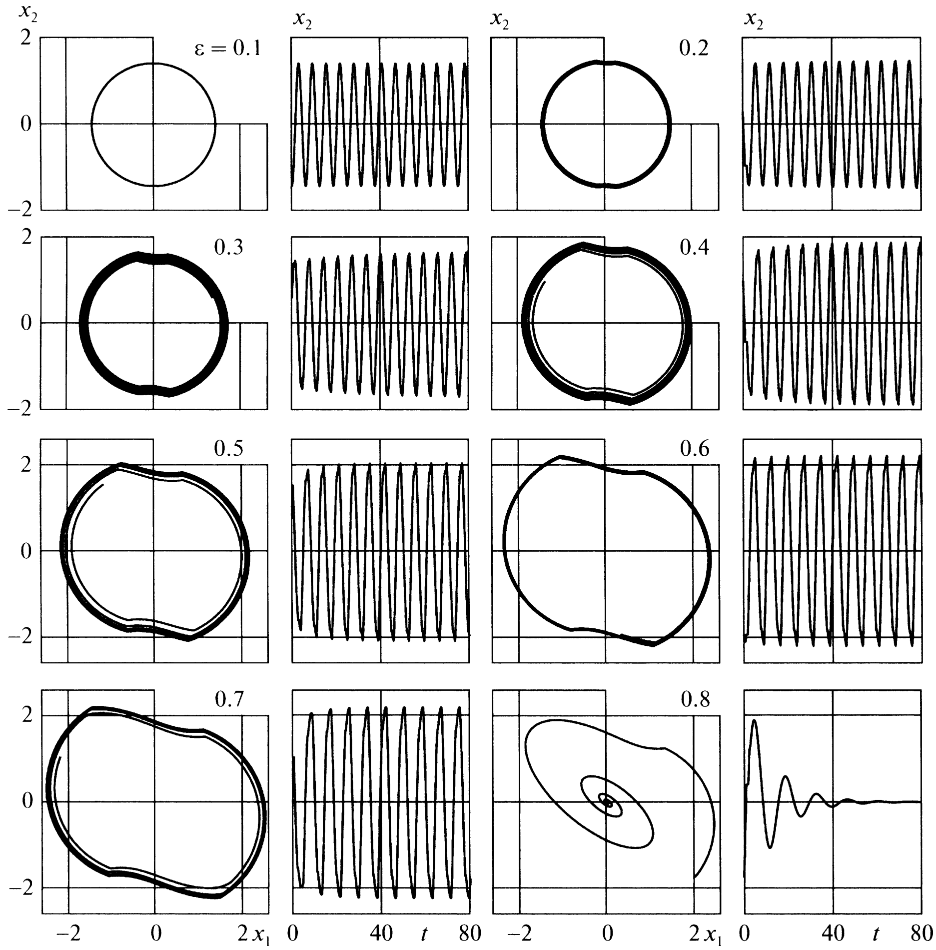

Fig. 16.

Here,

$$r^*q = -2, \quad \omega_0 = 1, \quad b_1 = -1, \quad b_2 = -1, \quad A = -1, \quad b = -1, \quad c = 1$$

The stability of the linear system (4.7) with $\varphi_j(\sigma) = k\sigma$ holds here for all $k \in (0, +\infty)$.

Suppose

$$\varphi_j(\sigma) = \begin{cases} \mu\sigma, & \forall\sigma \in (-\varepsilon_j, \varepsilon_j) \\ M\varepsilon_j^3, & \forall\sigma > \varepsilon_j \\ -M\varepsilon_j^3, & \forall\sigma < -\varepsilon_j \end{cases}$$

Here,

$$\mu = 2, \quad M = 1, \quad \varepsilon_1 = \varepsilon, \quad \varepsilon_2 = 0.1, \quad \varepsilon_3 = 0.2, \ldots, \quad \varepsilon_9 = 0.8$$

and ε is a small positive parameter.

According to Theorem 8, the initial data of the stable periodic oscillation in the first step $j = 1$ take the form

$$x_1(0) = O(\varepsilon), \quad x_3(0) = O(\varepsilon), \quad x_2(0) = -\sqrt{2} + O(\varepsilon)$$

Hence, when $j = 2$, we emit a trajectory from the point

$$x_1(0) = x_3(0) = 0, \quad x_2(0) = -\sqrt{2}$$

The projection of this trajectory onto the (x_1x_2) plane and its coordinate $x_2(t)$ are shown in Fig. 16. It is clear that, after the transition process, a stable periodic solution is reached.

Continuing this procedure when $j = 3, .., 8$, we find the periodic solutions, and, when $\varepsilon_9 = 0.8$, disappearance of the periodic solution and attraction to the stable equilibrium state is observed (Fig. 16).

Note that, when $\varepsilon_j = \sqrt{2}$, the non-linearity $\varphi_j(\sigma)$ is monotonic. The fact that there are no periodic solutions in the case of system (4.7) with such a non-linearity when $n = 3$ is well known.[63,64] Hence, here also, a stability loss bifurcation is observed and disappearance of a periodic solution.

The search for higher order systems, in which the algorithm for finding the periodic solutions leads to system (4.1) with a non-linearity $\psi(\sigma)$ from the Hurwitz sector, which is monotonic with respect to the two boundaries of the Hurwitz sector and has a stable periodic solution, is an important problem.

In the following example, we will show that the condition $\mu\beta * q \le \alpha\omega_0^2$ is a necessary condition and that the inequality

$$\mu\beta r^* < \alpha\omega_0^2 \tag{9.4}$$

is a sufficient condition for the absolute stability of system (4.6) when $n = 3$.

Example 5. *We now consider a third order system (4.6) with a transfer function of the form*

$$W(p) = \frac{\alpha p + \beta}{p^2 + \omega_0^2} + \frac{\varkappa}{p + \gamma}, \quad \omega_0 > 0, \quad \gamma > 0$$

The relations $\alpha > 0$ or $\alpha = 0$, $\kappa\beta > 0$ are necessary and sufficient conditions in order that system (4.6) in the case of $\varphi_0(\sigma) = k\sigma$ and small positive k should be stable as a whole. These are the so-called conditions of limit stability.[65]

When $\alpha > 0$, we make use of Popov's frequency criterion

$$\frac{1}{\mu} + \mathrm{Re}(1 + \theta i\omega)W(i\omega) > 0 \tag{9.5}$$

and, when $\alpha = 0$, $\kappa\beta > 0$, we make us of the criterion[63,64]

$$\mathrm{Re}[(i\omega)W(i\omega)] \neq 0, \quad \forall\omega \neq 0 \tag{9.6}$$

It is clear that inequality (9.6) is satisfied when $\alpha = 0$, $\kappa\beta > 0$. In this case, $(0, +\infty)$ is the sector of absolute stability. Inequality (9.5) takes the form

$$\frac{1}{\mu} + \frac{\beta}{\omega_0^2} + \frac{\gamma + \theta\omega^2}{\omega^2 + \gamma^2}\varkappa > 0, \quad \theta = \frac{\beta}{\alpha\omega_0^2} \tag{9.7}$$

Since the factor with κ is a monotonic function, inequality (9.7) is satisfied for the following conditions

$$\frac{1}{\mu} + \frac{\beta}{\omega_0^2} + \frac{\varkappa}{\gamma} > 0, \quad \frac{1}{\mu} + \frac{\beta}{\omega_0^2} + \frac{\varkappa\beta}{\alpha\omega_0^2} > 0$$

If

$$\frac{\varkappa}{\gamma} < \frac{\varkappa\beta}{\alpha\omega_0^2}$$

then absolute stability holds in the sector $(0, +\infty)$ when

$$\frac{\beta}{\omega_0^2} + \frac{\varkappa}{\gamma} > 0$$

and in the sector

$$\left(0, -\left(\frac{\beta}{\omega_0^2} + \frac{\varkappa}{\gamma}\right)^{-1}\right)$$

when

$$\frac{\beta}{\omega_0^2} + \frac{\varkappa}{\gamma} < 0$$

This sector is identical to the maximum Hurwitz sector, and, consequently, the Aizerman hypothesis holds in this case.

If

$$\frac{\varkappa}{\gamma} > \frac{\varkappa\beta}{\alpha\omega_0^2} \tag{9.8}$$

then absolute stability holds when

$$\mu\beta(\alpha + \varkappa) + \alpha\omega_0^2 > 0 \tag{9.9}$$

Condition (9.9) is identical with the inequality (9.7). Hence, the condition $\mu\beta r * q \leq \alpha\omega_0^2$ is a necessary condition and (9.4) is a sufficient condition for absolute stability of system (4.3) when $n = 3$. It follows from this that, if inequality (9.8) is satisfied, the Aizerman hypothesis holds.

A book [13] is devoted to such an analysis of system (4.3) with $n = 3$.

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